# Week 10 – Indeterminate Forms, Optimization

**Indeterminate Forms:** Some limits like

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty-\infty,\quad 0^0,\quad 1^\infty,\quad \infty^0$$

are called indeterminate forms. These limit may turn out to be definite numbers, or infinity, or may not exist.

Infinity is not a number. For example, $1^\infty$ is a shorthand notation for

$$\lim_{x\to a} \big(f(x)\big)^{g(x)}$$

where

$$\lim_{x\to a} f(x) = 1 \quad \text{and} \quad \lim_{x\to a} g(x) = \infty$$

(It does NOT mean $1\cdot 1\cdot 1\cdots$)

**L'Hôpital's Rule:** Consider

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

Assume that we have an indeterminate form of the type

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

Suppose $g'(x) \neq 0$ on an open interval containing $a$ (except possibly at $x = a$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if this limit exists, or is $\pm\infty$.

**Exercise 10-1:** Evaluate the limit

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2}$$

**Solution:** We can solve this using multiplication by $\cos x + 1$, but this is in the form $\frac{0}{0}$ and L'Hôpital's Rule gives the same result with less effort:

$$\begin{aligned}\lim_{x\to 0} \frac{\cos x - 1}{x^2} &= \lim_{x\to 0} \frac{-\sin x}{2x} \\ &= -\frac{1}{2}\end{aligned}$$

**Exercise 10-2:** Evaluate the limit

$$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$

**Solution:**

$$\frac{0}{0} \quad \Rightarrow \quad \text{use L'Hôpital}$$

Sometimes we need to use it more than once.

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0} \frac{1-\cos x}{3x^2} \\ &= \lim_{x\to 0} \frac{\sin x}{6x} \\ &= \lim_{x\to 0} \frac{\cos x}{6} \\ &= \lim_{x\to 0} \frac{1}{6}\end{aligned}$$

**Exercise 10-3:** Evaluate the limit

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

**Solution:**

$$\frac{\infty}{\infty} \quad \Rightarrow \quad \text{use L'Hôpital}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x}{x^3} &= \lim_{x\to\infty} \frac{e^x}{3x^2} \\ &= \lim_{x\to\infty} \frac{e^x}{6x} \\ &= \lim_{x\to\infty} \frac{e^x}{6} \\ &= \infty\end{aligned}$$

Exponential function increases faster than all polynomials.

**Exercise 10-4:** Evaluate the following limits using use L'Hôpital's rule:

**a)** $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$

**b)** $\lim_{x\to 0} \dfrac{1-\cos x}{2x-x^2}$

**c)** $\lim_{x\to 3} \dfrac{x^3-4x-15}{x^2+x-12}$

**d)** $\lim_{\theta\to \frac{\pi}{2}} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

**e)** $\lim_{x\to \infty} \dfrac{\ln x}{\sqrt[3]{x}}$

**Other Indeterminate Forms:** We can usually transform $\infty - \infty$ or $1^\infty$ into $0/0$ using logarithms or arithmetic operations.

**Exercise 10-5:** Evaluate the limit

$$\lim_{x\to 1^+} \frac{1}{x-1} - \frac{1}{\ln x}$$

**Solution:**

$$\lim_{x\to 1^+} \frac{1}{x-1} - \frac{1}{\ln x} = \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x}$$

This is in the form $\dfrac{0}{0} \quad \Rightarrow \quad$ use L'Hôpital

$$\lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{\ln x + 1 - \frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x\ln x + x - 1}$$

$$= \lim_{x\to 1^+} \frac{-1}{\ln x + 1 + 1}$$

$$= -\frac{1}{2}$$

**Exercise 10-6:** Evaluate the limit

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

**Solution:** This is in the form $1^\infty \quad \Rightarrow \quad$ find logarithm, then use L'Hôpital.

Let: $L = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Then: $\ln L = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

This is in the form: $\dfrac{0}{0} \quad \Rightarrow \quad$ use L'Hôpital

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

$$= 1$$

$$\ln L = 1 \quad \Rightarrow \quad L = e$$

In other words

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

**Exercise 10-7:** Evaluate the following limits:

**a)** $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x$

**b)** $\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^x$

**c)** $\lim_{x\to\infty} \left(\frac{x-2}{x+3}\right)^{4x}$

**d)** $\lim_{x \to 1} (1 - x) \tan\left(\frac{\pi x}{2}\right)$

**e)** $\lim_{x \to 0^+} (\sin x)^x$

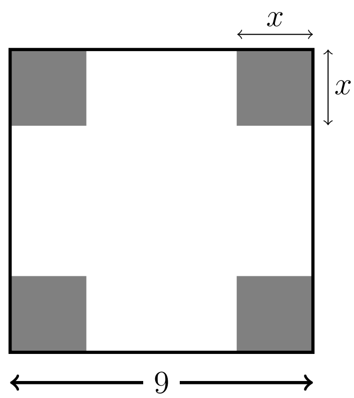
**f)** $\lim_{x \to \infty} x^{1/x}$

**g)** $\lim_{x \to \infty} x^{1/\ln x}$

**Applied Optimization:** Finding the maximum or minimum of a function has many real-life applications. For these problems:

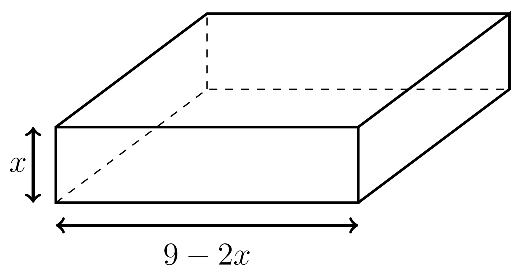
- Express the quantity to be maximized or minimized as a function of the independent variable. (We will call it $x$)
- Determine the interval over which $x$ changes.
- Solve the problem in the usual way. (Find the critical points, check the function at critical points and endpoints)

**Exercise 10-8:** A piece of cardboard is shaped as a $9 \times 9$ square. We will cut four small squares from the corners and make an open top box. What is the maximum possible volume of the box?




**Solution:** If the squares have edge length $x$, we can express the volume as:

$$V(x) = x(9 - 2x)^2 = 81x - 36x^2 + 4x^3$$



Considering the maximum and minimum possible values, we can see that

$$x \in [0, 9/2]$$

Now we can use maximization procedure:

$$V'(x) = 81 - 72x + 12x^2 = 0$$

$$27 - 24x + 4x^2 = 0$$

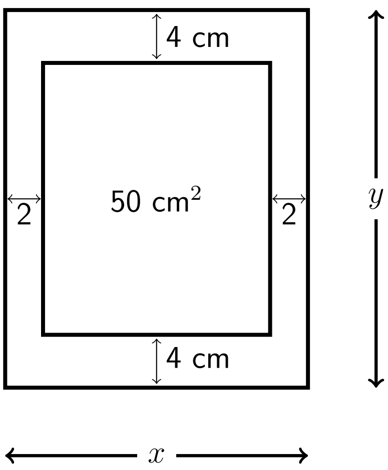
$$(2x-9)(2x-3) = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Checking all critical and endpoints, we find that $x = \frac{3}{2}$ gives the maximum volume, which is:

$$V = 54$$

**Exercise 10-9:** You are designing a rectangular poster to contain 50 $\text{cm}^2$ of picture area with a 4 cm margin at the top and bottom and a 2 cm margin at each side. Find the dimensions $x$ and $y$ that will minimize the total area of the poster.




**Solution:**

$$(x-4)(y-8) = 50 \quad \Rightarrow \quad y = \frac{50}{x-4} + 8$$

$$A = xy = x\left(\frac{50}{x-4} + 8\right)$$

$$A' = \frac{-200}{(x-4)^2} + 8 = 0$$

$$(x-4)^2 = 25$$

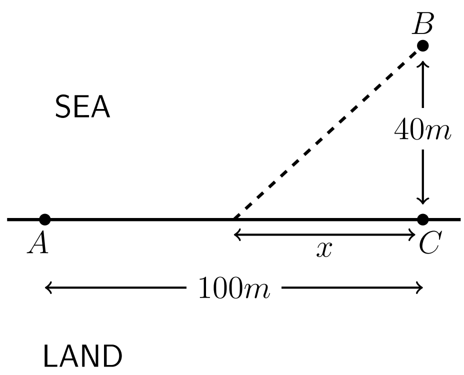
$$\Rightarrow \quad x = 9 \quad \text{and} \quad y = 18$$

**Exercise 10-10:** A piece of wire of length 80 is cut into two pieces and each one is made into a square. How should this be done to minimize the total area?

**Exercise 10-11:** Find the dimensions of the right circular cylinder of the greatest volume if the surface area is $54\pi$.

**Answer:** $r = 3, h = 6$

**Exercise 10-12:** A swimmer is drowning on point $B$. You are at point $A$. You may run up to point $C$ and then swim, or you may start swimming a distance $x$ earlier. Your running speed is 10 m/s and your swimming speed is 6 m/s. What is the ideal $x$?



## Review Exercises

**Exercise 10-13:** Evaluate the following limits using use L'Hôpital's rule:

**a)** $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{\tan(7x)}$

**b)** $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{\sin^2 x}$

**c)** $\lim\limits_{x \to 1} \dfrac{x^6 - 1}{x^4 - 1}$

**d)** $\lim\limits_{x \to 0} \dfrac{\sqrt{a + bx} - \sqrt{a + cx}}{x}$

**Exercise 10-14:** Evaluate the following limits:

**a)** $\lim\limits_{x \to \infty} \left( \dfrac{1}{1 + 3x} \right)^{4x}$

**b)** $\lim\limits_{x \to \infty} \; x^{1/x}$

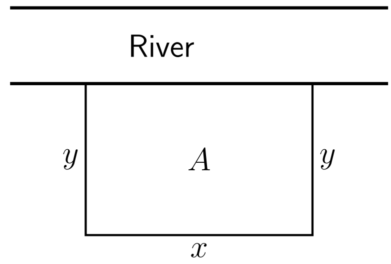
**c)** $\lim\limits_{x \to 0^+} \; x \ln x$

**d)** $\lim\limits_{x \to 0^+} \; (1 + \sin 3x)^{\csc 3x}$

**e)** $\lim\limits_{x \to 0^+} \; \dfrac{1}{x} - \dfrac{1}{e^x - 1}$

**f)** $\lim\limits_{x \to \infty} \; x \sin\left( \dfrac{\pi}{2 + 7x} \right)$

**Exercise 10-15:** We will cover a rectangular area with a 360m-long fence. The area is near a river so we will only cover the three sides. Find the maximum possible area.




$$y + x + y = 360$$

**Exercise 10-16:** What is the maximum possible area of the rectangle with its base on the $x-$axis and its two upper vertices are on the graph of $y = 4 - x^2$?

**Exercise 10-17:** Find the shortest distance between the point $(2, 0)$ and the curve $y = \sqrt{x}$

**Exercise 10-18:** You are selling tickets for a concert. If the price of a ticket is $\$1.50$, you sell 600 tickets. Market research reveals that, for each 5c decrease, sales will increase by 40 and for each 5c increase, decrease by 40. For example, at $\$1.45$ you will sell 640 tickets.

What should the ticket price be for largest possible revenue?

**Answer:** $1.125$

## — End of WEEK —

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