

# Week 10 – Indeterminate Forms, Optimization

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**Indeterminate Forms:** Some limits like

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

are called indeterminate forms. These limit may turn out to be definite numbers, or infinity, or may not exist.

Infinity is not a number. For example,  $1^\infty$  is a shorthand notation for

$$\lim_{x \rightarrow a} (f(x))^{g(x)}$$

where

$$\lim_{x \rightarrow a} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \infty$$

(It does NOT mean  $1 \cdot 1 \cdot 1 \cdots$ )

**L'Hôpital's Rule:** Consider

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

Assume that we have an indeterminate form of the type

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

Suppose  $g'(x) \neq 0$  on an open interval containing  $a$  (except possibly at  $x = a$ ). Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if this limit exists, or is  $\pm\infty$ .

**Exercise 10-1:** Evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$$

**Solution:** We can solve this using multiplication by  $\cos x + 1$ , but this is in the form  $\frac{0}{0}$  and L'Hôpital's Rule gives the same result with less effort:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{-\sin x}{2x} \\ &= -\frac{1}{2} \end{aligned}$$

**Exercise 10-2:** Evaluate the limit

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

**Solution:**

$$\frac{0}{0} \Rightarrow \text{use L'Hôpital}$$

Sometimes we need to use it more than once.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{6} \\ &= \lim_{x \rightarrow 0} \frac{1}{6} \end{aligned}$$

**Exercise 10-3:** Evaluate the limit

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3}$$

**Solution:**

$$\frac{\infty}{\infty} \Rightarrow \text{use L'Hôpital}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^x}{x^3} &= \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{6x} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{6} \\ &= \infty \end{aligned}$$

Exponential function increases faster than all polynomials.

**Exercise 10-4:** Evaluate the following limits using use L'Hôpital's rule:

a)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

b)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x - x^2}$

c)  $\lim_{x \rightarrow 3} \frac{x^3 - 4x - 15}{x^2 + x - 12}$

d)  $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1 - \sin \theta}{1 + \cos 2\theta}$

e)  $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$

**Other Indeterminate Forms:** We can usually transform  $\infty - \infty$  or  $1^\infty$  into  $0/0$  using logarithms or arithmetic operations.

**Exercise 10-5:** Evaluate the limit

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} - \frac{1}{\ln x}$$

**Solution:**

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} - \frac{1}{\ln x} = \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1) \ln x}$$

This is in the form  $\frac{0}{0} \Rightarrow$  use L'Hôpital

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1) \ln x} &= \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln x + 1 - \frac{1}{x}} \\ &= \lim_{x \rightarrow 1^+} \frac{1-x}{x \ln x + x - 1} \\ &= \lim_{x \rightarrow 1^+} \frac{-1}{\ln x + 1 + 1} \\ &= -\frac{1}{2} \end{aligned}$$

**Exercise 10-6:** Evaluate the limit

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

**Solution:** This is in the form  $1^\infty \Rightarrow$  find logarithm, then use L'Hôpital.

$$\text{Let: } L = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$\text{Then: } \ln L = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right)$$

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

This is in the form:  $\frac{0}{0} \Rightarrow$  use L'Hôpital

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{\left(1 + \frac{1}{x}\right)} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} \\ &= 1 \end{aligned}$$

$$\ln L = 1 \Rightarrow L = e$$

In other words

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

**Exercise 10-7:** Evaluate the following limits:

a)  $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x$

b)  $\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x}\right)^x$

c)  $\lim_{x \rightarrow \infty} \left(\frac{x-2}{x+3}\right)^{4x}$

d)  $\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

e)  $\lim_{x \rightarrow 0^+} (\sin x)^x$

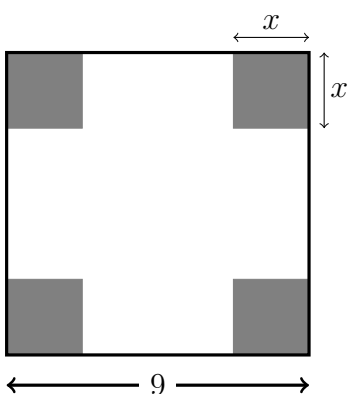
f)  $\lim_{x \rightarrow \infty} x^{1/x}$

g)  $\lim_{x \rightarrow \infty} x^{1/\ln x}$

**Applied Optimization:** Finding the maximum or minimum of a function has many real-life applications. For these problems:

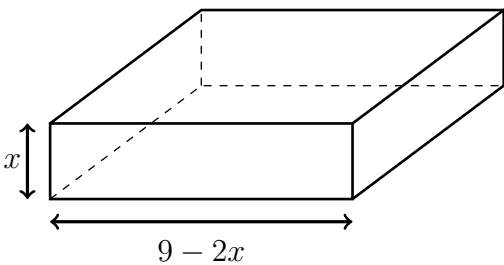
- Express the quantity to be maximized or minimized as a function of the independent variable. (We will call it  $x$ )
- Determine the interval over which  $x$  changes.
- Solve the problem in the usual way. (Find the critical points, check the function at critical points and endpoints)

**Exercise 10-8:** A piece of cardboard is shaped as a  $9 \times 9$  square. We will cut four small squares from the corners and make an open top box. What is the maximum possible volume of the box?



**Solution:** If the squares have edge length  $x$ , we can express the volume as:

$$V(x) = x(9 - 2x)^2 = 81x - 36x^2 + 4x^3$$



Considering the maximum and minimum possible values, we can see that

$$x \in [0, 9/2]$$

Now we can use maximization procedure:

$$V'(x) = 81 - 72x + 12x^2 = 0$$

$$27 - 24x + 4x^2 = 0$$

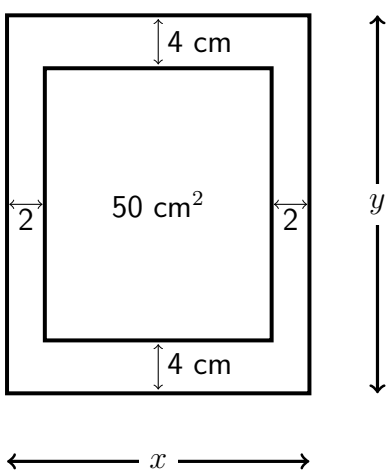
$$(2x - 9)(2x - 3) = 0$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Checking all critical and endpoints, we find that  $x = \frac{3}{2}$  gives the maximum volume, which is:

$$V = 54$$

**Exercise 10-9:** You are designing a rectangular poster to contain  $50 \text{ cm}^2$  of picture area with a 4 cm margin at the top and bottom and a 2 cm margin at each side. Find the dimensions  $x$  and  $y$  that will minimize the total area of the poster.



**Solution:**

$$(x - 4)(y - 8) = 50 \Rightarrow y = \frac{50}{x - 4} + 8$$

$$A = xy = x \left( \frac{50}{x - 4} + 8 \right)$$

$$A' = \frac{-200}{(x - 4)^2} + 8 = 0$$

$$(x - 4)^2 = 25$$

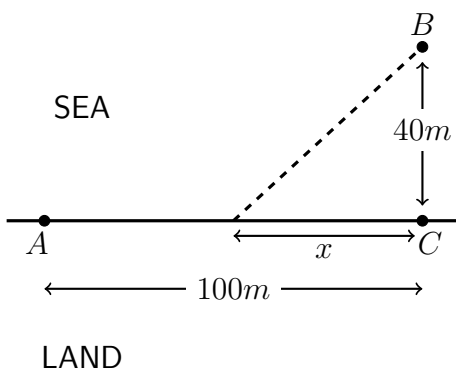
$$\Rightarrow x = 9 \quad \text{and} \quad y = 18$$

**Exercise 10-10:** A piece of wire of length 80 is cut into two pieces and each one is made into a square. How should this be done to minimize the total area?

**Exercise 10-11:** Find the dimensions of the right circular cylinder of the greatest volume if the surface area is  $54\pi$ .

**Answer:**  $r = 3, h = 6$

**Exercise 10-12:** A swimmer is drowning on point  $B$ . You are at point  $A$ . You may run up to point  $C$  and then swim, or you may start swimming a distance  $x$  earlier. Your running speed is 10 m/s and your swimming speed is 6 m/s. What is the ideal  $x$ ?



## Review Exercises

**Exercise 10-13:** Evaluate the following limits using L'Hôpital's rule:

a)  $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(7x)}$

b)  $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{\sin^2 x}$

c)  $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^4 - 1}$

d)  $\lim_{x \rightarrow 0} \frac{\sqrt{a + bx} - \sqrt{a + cx}}{x}$

**Exercise 10-14:** Evaluate the following limits:

a)  $\lim_{x \rightarrow \infty} \left( \frac{1}{1 + 3x} \right)^{4x}$

b)  $\lim_{x \rightarrow \infty} x^{1/x}$

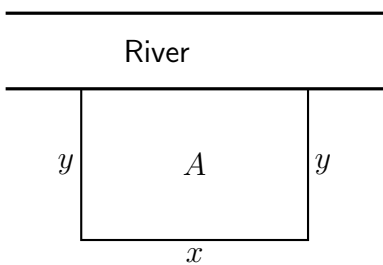
c)  $\lim_{x \rightarrow 0^+} x \ln x$

d)  $\lim_{x \rightarrow 0^+} (1 + \sin 3x)^{\csc 3x}$

e)  $\lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{e^x - 1}$

f)  $\lim_{x \rightarrow \infty} x \sin \left( \frac{\pi}{2 + 7x} \right)$

**Exercise 10-15:** We will cover a rectangular area with a 360m-long fence. The area is near a river so we will only cover the three sides. Find the maximum possible area.



$$y + x + y = 360$$

**Exercise 10-16:** What is the maximum possible area of the rectangle with its base on the  $x$ -axis and its two upper vertices are on the graph of  $y = 4 - x^2$ ?

**Exercise 10-17:** Find the shortest distance between the point  $(2, 0)$  and the curve  $y = \sqrt{x}$

**Exercise 10-18:** You are selling tickets for a concert. If the price of a ticket is \$1.50, you sell 600 tickets. Market research reveals that, for each 5c decrease, sales will increase by 40 and for each 5c increase, decrease by 40. For example, at \$1.45 you will sell 640 tickets.

What should the ticket price be for largest possible revenue?

**Answer:** 1.125

— End of WEEK —

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