Week 11 – The Integral

Antiderivative: If f is the derivative of F, then F is the antiderivative of f. i.e. $F' = f$

For example, the antiderivative of $f = x$ is $F = \frac{x^2}{2}$, or $F = \frac{x^2}{2} + 5$ or $F = \frac{x^2}{2} - 7$.

- If F is the antiderivative of f, then every antiderivative of f has the form $F + c$ (where c is an arbitrary constant)
- The collection of all antiderivatives of f is called the indefinite integral of f.

$$\int f(x)\,dx = F(x) + c$$

Using the fact that integral and derivative are inverse operations, we obtain:

$$\begin{aligned}
\int 1\,dx &= x + c \\
\int x\,dx &= \frac{x^2}{2} + c \\
\int x^k\,dx &= \frac{x^{k+1}}{k+1} + c, \quad k \neq -1 \\
\int \frac{1}{x}\,dx &= \ln|x| + c \\
\int \cos x\,dx &= \sin x + c \\
\int \sin x\,dx &= -\cos x + c \\
\int e^x\,dx &= e^x + c \\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \arcsin x + c
\end{aligned}$$

Exercise 11-1: Evaluate the following integrals:

a) $\int \left(\frac{1}{x^3} - 2x + 4\right)\,dx$

b) $\int \frac{1}{x^2}\,dx$

c) $\int e^x + 1\,dx$

d) $\int \frac{3 - 2x + x^2}{x}\,dx$

e) $\int e^{-x}\,dx$

Summation Notation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

For example:

$$\sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = \sum_{i=0}^{3} (i+1)^2$$

Exercise 11-2: Express the following using sigma notation:

a) $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} \cdots + \frac{1}{100}$

b) $1 + 2 + 4 + 8 + 16 + 32 + 64$

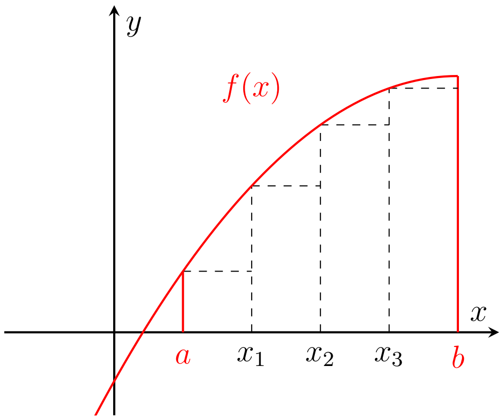
c) $x + x^3 + x^5 + \cdots + x^{17}$

Area Under a Curve: We can approximate the area under the graph of f between $x = a$ and $x = b$ using rectangles. If we divide the interval $[a, b]$ into n parts, such that

$$\begin{aligned}
x_0 &= a, \\
x_1 &= a + \Delta x, \\
x_2 &= a + 2\Delta x, \\
\vdots &= \vdots \\
x_n &= a + n\Delta x = b
\end{aligned}$$

and assuming f is increasing, we obtain:

$$\sum_{i=0}^{n-1} f(x_i)\Delta x \leqslant A \leqslant \sum_{i=1}^{n} f(x_i)\Delta x$$



Rather than choosing the right or left endpoint, we can choose any number inside the interval to estimate average $f(x)$ on that interval. Such a generalization gives:

Riemann Sum: Let f be a function defined on $[a, b]$. Let us divide $[a, b]$ into n parts, and let us choose a point x_i^* from each interval. Then, Riemann sum for f is

$$R = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Definite Integral: The definite integral of the function f from a to b is the number

$$I = \int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. Then we say f is integrable on $[a, b]$.

Theorem: If the function f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Exercise 11-3: Use Riemann sum to compute the following:

a) $\displaystyle\int_a^b 1\,dx$

b) $\displaystyle\int_a^b x\,dx$

Definite Integral Properties:

$$\int_a^a f(x)dx = 0$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\text{Min} f \cdot (b-a) \leqslant \int_a^b f(x)dx \leqslant \text{Max} f \cdot (b-a)$$

To use the Riemann sums to evaluate integrals is almost impossible except for the simplest functions. Instead, we use the following theorem:

The Fundamental Theorem of Calculus: Let f be a continuous function on the interval $[a, b]$. Then

- $F(x) = \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of f, i.e. $F' = f$
- If F is any anti-derivative of f, then
$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Exercise 11-4: Evaluate $\displaystyle\int_1^9 \frac{5}{\sqrt{x}}\,dx$

Solution:

$$\begin{aligned}
\int_1^9 \frac{5}{\sqrt{x}}\,dx &= \int_1^9 5x^{-1/2}\,dx \\
&= \frac{5\,x^{1/2}}{1/2}\Bigg|_1^9 \\
&= 5 \cdot 2 \cdot 9^{1/2} - 5 \cdot 2 \cdot 1^{1/2} \\
&= 30 - 10 \\
&= 20
\end{aligned}$$

Exercise 11-5: Evaluate the following definite integrals:

a) $\int_0^1 x^5\,dx$

b) $\int_3^7 \frac{1}{x}\,dx$

c) $\int_1^3 \frac{x^5+x}{x^4}\,dx$

d) $\int_1^8 x^{2/3}\,dx$

e) $\int_0^{\pi/4} \sec^2\theta\,d\theta$

f) $\int_0^{\pi/6} \sec\theta\,\tan\theta\,d\theta$

g) $\int_0^{1/2} \frac{3}{\sqrt{1-u^2}}\,du$

h) $\int_0^1 \frac{1}{1+x^2}\,dx$

Derivative of an Integral: Using the fundamental theorem and the chain rule, we can easily show that

$$\frac{d}{dx}\int_u^v f(t)\,dt = f(v)\,v' - f(u)\,u'$$

where

$$u = u(x), \quad v = v(x)$$

Exercise 11-6: Find

$$\frac{d}{dx}\int_1^{x^5} \cos t\,dt$$

Solution: We do not have to evaluate the integral. Using the above formula:

$$\begin{aligned}\frac{d}{dx}\int_1^{x^5} \cos t\,dt &= \frac{d}{du}\int_1^{u} \cos t\,dt \cdot \frac{du}{dt} \\ &= \cos u \cdot \frac{du}{dt} \\ &= \cos(x^5)\,5x^4\end{aligned}$$

Exercise 11-7: Find $\frac{dy}{dx}$ for the following functions:

a) $y = \int_0^x \sqrt{1+t^2}\,dt$

b) $y = \int_0^{3x-5} \tan^2 t\,dt$

c) $y = \int_x^{2x^2} e^{t^2}\,dt$

d) $y = \int_\pi^{\sin x} \ln(1+t+t^2)\,dt$

Review Exercises

Exercise 11-8: Evaluate the following integrals:

a) $\int_0^1 (2x-1)(x+3), dx$

b) $\int_1^2 \frac{t^5-5t^2-t}{t^2}\,dt$

c) $\int_{1/\sqrt{3}}^1 \frac{4}{1+u^2}\,du$

d) $\int_0^9 2^x\,dx$

e) $\int_1^e 1-\frac{2}{x}\,dx$

f) $\displaystyle\int_1^8 \frac{1}{\sqrt[3]{s}}\,ds$

g) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2\theta\,d\theta$

Exercise 11-9: Find $\dfrac{dy}{dx}$:

a) $\displaystyle y = \int_0^{x^8} \sqrt{t}\,dt$

b) $\displaystyle y = \int_0^{x^3} \sin t^2\,dt$

c) $\displaystyle y = \int_1^{\sin x} (1-t^2)^4\,dt$

d) $\displaystyle y = \int_0^{e^x} \frac{1}{1+t^2}\,dt$

e) $\displaystyle y = \int_{2x^3}^{5} \left(\ln t + e^t\right)\,dt$

— End of WEEK —

Author: Dr. Emre Sermutlu

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