

# Week 11 – The Integral

**Antiderivative:** If  $f$  is the derivative of  $F$ , then  $F$  is the antiderivative of  $f$ . i.e.  $F' = f$

For example, the antiderivative of  $f = x$  is  $F = \frac{x^2}{2}$ , or  $F = \frac{x^2}{2} + 5$  or  $F = \frac{x^2}{2} - 7$ .

- If  $F$  is the antiderivative of  $f$ , then every antiderivative of  $f$  has the form  $F + c$  (where  $c$  is an arbitrary constant)
- The collection of all antiderivatives of  $f$  is called the indefinite integral of  $f$ .

$$\int f(x) dx = F(x) + c$$

Using the fact that integral and derivative are inverse operations, we obtain:

$$\int 1 dx = x + c$$

$$\int x dx = \frac{x^2}{2} + c$$

$$\int x^k dx = \frac{x^{k+1}}{k+1} + c, \quad k \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int e^x dx = e^x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$$

**Exercise 11-1:** Evaluate the following integrals:

a)  $\int \left( \frac{1}{x^3} - 2x + 4 \right) dx$

b)  $\int \frac{1}{x^2} dx$

c)  $\int e^x + 1 dx$

d)  $\int \frac{3 - 2x + x^2}{x} dx$

e)  $\int e^{-x} dx$

**Summation Notation:**

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

For example:

$$\sum_{i=1}^4 i^2 = 1 + 4 + 9 + 16 = \sum_{i=0}^3 (i+1)^2$$

**Exercise 11-2:** Express the following using sigma notation:

a)  $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} \cdots + \frac{1}{100}$

b)  $1 + 2 + 4 + 8 + 16 + 32 + 64$

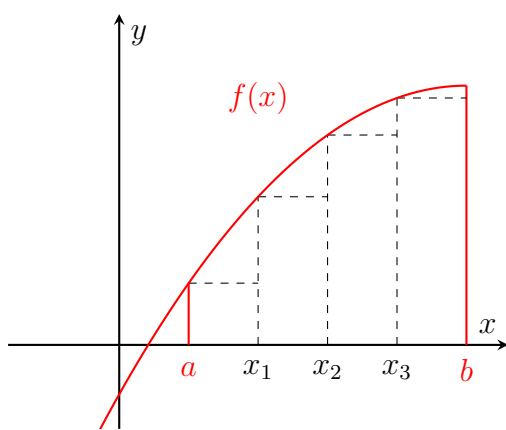
c)  $x + x^3 + x^5 + \cdots + x^{17}$

**Area Under a Curve:** We can approximate the area under the graph of  $f$  between  $x = a$  and  $x = b$  using rectangles. If we divide the interval  $[a, b]$  into  $n$  parts, such that

$$\begin{aligned} x_0 &= a, \\ x_1 &= a + \Delta x, \\ x_2 &= a + 2\Delta x, \\ &\vdots \\ x_n &= a + n\Delta x = b \end{aligned}$$

and assuming  $f$  is increasing, we obtain:

$$\sum_{i=0}^{n-1} f(x_i)\Delta x \leq A \leq \sum_{i=1}^n f(x_i)\Delta x$$



Rather than choosing the right or left endpoint, we can choose any number inside the interval to estimate average  $f(x)$  on that interval. Such a generalization gives:

**Riemann Sum:** Let  $f$  be a function defined on  $[a, b]$ . Let us divide  $[a, b]$  into  $n$  parts, and let us choose a point  $x_i^*$  from each interval. Then, Riemann sum for  $f$  is

$$R = \sum_{i=1}^n f(x_i^*)\Delta x_i$$

**Definite Integral:** The definite integral of the function  $f$  from  $a$  to  $b$  is the number

$$I = \int_a^b f(x)dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$$

if this limit exists. Then we say  $f$  is integrable on  $[a, b]$ .

**Theorem:** If the function  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .

**Exercise 11-3:** Use Riemann sum to compute the following:

a)  $\int_a^b 1 dx$

$$\text{b) } \int_a^b x \, dx$$

### Definite Integral Properties:

$$\int_a^a f(x) \, dx = 0$$

$$\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$$

$$\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

$$\text{Min} f \cdot (b - a) \leq \int_a^b f(x) \, dx \leq \text{Max} f \cdot (b - a)$$

To use the Riemann sums to evaluate integrals is almost impossible except for the simplest functions. Instead, we use the following theorem:

### The Fundamental Theorem of Calculus:

Let  $f$  be a continuous function on the interval  $[a, b]$ . Then

- $F(x) = \int_a^x f(t) \, dt$  is an antiderivative of  $f$ , i.e.  $F' = f$
- If  $F$  is any anti-derivative of  $f$ , then

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

**Exercise 11-4:** Evaluate  $\int_1^9 \frac{5}{\sqrt{x}} \, dx$

### Solution:

$$\begin{aligned} \int_1^9 \frac{5}{\sqrt{x}} \, dx &= \int_1^9 5x^{-1/2} \, dx \\ &= \frac{5x^{1/2}}{1/2} \Big|_1^9 \\ &= 5 \cdot 2 \cdot 9^{1/2} - 5 \cdot 2 \cdot 1^{1/2} \\ &= 30 - 10 \\ &= 20 \end{aligned}$$

**Exercise 11-5:** Evaluate the following definite integrals:

$$\text{a) } \int_0^1 x^5 \, dx$$

$$\text{b) } \int_3^7 \frac{1}{x} \, dx$$

$$\text{c) } \int_1^3 \frac{x^5 + x}{x^4} dx$$

$$\text{d) } \int_1^8 x^{2/3} dx$$

$$\text{e) } \int_0^{\pi/4} \sec^2 \theta d\theta$$

$$\text{f) } \int_0^{\pi/6} \sec \theta \tan \theta d\theta$$

$$\text{g) } \int_0^{1/2} \frac{3}{\sqrt{1-u^2}} du$$

$$\text{h) } \int_0^1 \frac{1}{1+x^2} dx$$

**Derivative of an Integral:** Using the fundamental theorem and the chain rule, we can easily show that

$$\frac{d}{dx} \int_u^v f(t) dt = f(v) v' - f(u) u'$$

where

$$u = u(x), \quad v = v(x)$$

**Exercise 11-6:** Find

$$\frac{d}{dx} \int_1^{x^5} \cos t dt$$

**Solution:** We do not have to evaluate the integral. Using the fundamental theorem and chain rule:

$$\begin{aligned} \frac{d}{dx} \int_1^{x^5} \cos t dt &= \frac{d}{du} \int_1^u \cos t dt \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ &= \cos(x^5) 5x^4 \end{aligned}$$

**Exercise 11-7:** Find  $\frac{dy}{dx}$  for the following functions:

$$\text{a) } y = \int_0^x \sqrt{1+t^2} dt$$

$$\text{b) } y = \int_0^{3x-5} \tan^2 t dt$$

$$\text{c) } y = \int_x^{2x^2} e^{t^2} dt$$

$$\text{d) } y = \int_{\pi}^{\sin x} \ln(1+t+t^2) dt$$

## Review Exercises

**Exercise 11-8:** Evaluate the following integrals:

a)  $\int_0^1 (2x - 1)(x + 3), dx$

b)  $\int_1^2 \frac{t^5 - 5t^2 - t}{t^2} dt$

c)  $\int_{1/\sqrt{3}}^1 \frac{4}{1 + u^2} du$

d)  $\int_0^9 2^x dx$

e)  $\int_1^e 1 - \frac{2}{x} dx$

f)  $\int_1^8 \frac{1}{\sqrt[3]{s}} ds$

g)  $\int_{-\pi/4}^{\pi/4} \sec^2 \theta d\theta$

**Exercise 11-9:** Find  $\frac{dy}{dx}$ :

a)  $y = \int_0^{x^8} \sqrt{t} dt$

b)  $y = \int_0^{x^3} \sin t^2 dt$

c)  $y = \int_1^{\sin x} (1 - t^2)^4 dt$

d)  $y = \int_0^{e^x} \frac{1}{1 + t^2} dt$

e)  $y = \int_{2x^3}^5 (\ln t + e^t) dt$

— End of WEEK —