Week 12 – Substitution-Areas

Substitution: We know that

$$\frac{d}{dx}F(u(x)) = \frac{dF(u)}{du}\,\frac{du(x)}{dx}$$

by the chain rule. If we integrate both sides, we see that

$$\int f(u(x))\,u'(x)\,dx = F(u(x)) + c$$

where $f = F'$, or more simply

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

Exercise 12-1: Evaluate the integral

$$\int (x^4+1)^2\,4x^3\,dx$$

Solution: It is possible the expand the parenthesis, but there is no need.

$$u = x^4 + 1 \quad \Rightarrow \quad du = 4x^3\,dx$$

The new integral is:

$$\int u^2\,du = \frac{u^3}{3} + c$$

But we have to express this in terms of the original variable:

$$\frac{(x^4+1)^3}{3} + c$$

Exercise 12-2: Evaluate the integral

$$\int e^{3x^2}\,x\,dx$$

Solution: Let $I = \displaystyle\int e^{3x^2}\,x\,dx$

$$\begin{aligned} u = 3x^2 \quad &\Rightarrow \quad du = 6x\,dx \\ &\Rightarrow \quad x\,dx = \frac{1}{6}\,du \end{aligned}$$

$$\begin{aligned} I &= \int e^u \frac{1}{6}\,du \\ &= \frac{1}{6}e^u + c \\ &= \frac{e^{3x^2}}{6} + c \end{aligned}$$

Exercise 12-3: Evaluate the following integrals:

a) $\displaystyle\int (x^2+1)^{12} x\,dx$

b) $\displaystyle\int \frac{1}{(3x+2)^3}\,dx$

c) $\int \tan x \, dx$

d) $\int \sin 4x \, dx$

e) $\int \sqrt{13x+7} \, dx$

f) $\int (1-\cos x)^3 \sin x \, dx$

g) $\int \frac{x+3}{x^2+6x-1} \, dx$

h) $\int \frac{x+3}{(x^2+6x-1)^2} \, dx$

i) $\int \frac{1}{1+9x^2} \, dx$

j) $\int \frac{5}{16+25x^2} \, dx$

k) $\int \cos(7x^2+4) \, x \, dx$

Substitution in Definite Integrals: If u' is continuous on the interval $[a,b]$ and f is continuous on the range of u then

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

When using substitution in definite integrals, **don't** forget to transform the limits!

Exercise 12-4: Evaluate the definite integral

$$\int_0^1 8x(x^2+2)^3 \, dx$$

Solution: Using $u = x^2+2, \quad du = 2xdx$ and

$$x = 0 \quad \Rightarrow \quad u = 2,$$

$$x = 1 \quad \Rightarrow \quad u = 3$$

we obtain

$$\begin{aligned} I &= \int_2^3 4u^3 \, du \\ &= u^4 \Big|_2^3 \\ &= 81 - 16 \\ &= 65 \end{aligned}$$

Another idea is to evaluate it as an indefinite integral, rewrite u in terms of x and then use limits for x.
Once again, using $u = x^2+2, \quad du = 2xdx$

$$\begin{aligned} \int 8x(x^2+2)^3 \, dx &= \int 4u^3 \, du \\ &= u^4 + c \\ &= (x^2+2)^4 + c \end{aligned}$$

$$\begin{aligned} \int_0^1 8x(x^2+2)^3 \, dx &= (x^2+2)^4 \Big|_0^1 \\ &= 81 - 16 \\ &= 65 \end{aligned}$$

Exercise 12-5: Evaluate the following definite integrals.

a) $\displaystyle\int_0^4 x\sqrt{x^2+9}\,dx$

b) $\displaystyle\int_0^{\pi/2} \sin x\cos x\,dx$

c) $\displaystyle\int_0^8 x\sqrt{x+1}\,dx$

d) $\displaystyle\int_1^3 \frac{x+2}{x^2+4x+7}\,dx$

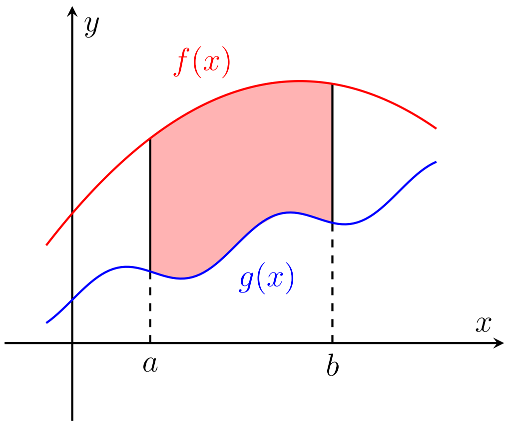
e) $\displaystyle\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x\,dx$

f) $\displaystyle\int_1^4 \frac{(1+\sqrt{x})^4}{\sqrt{x}}\,dx$

g) $\displaystyle\int_0^{\sqrt{\pi}} t\sin\frac{t^2}{2}\,dt$

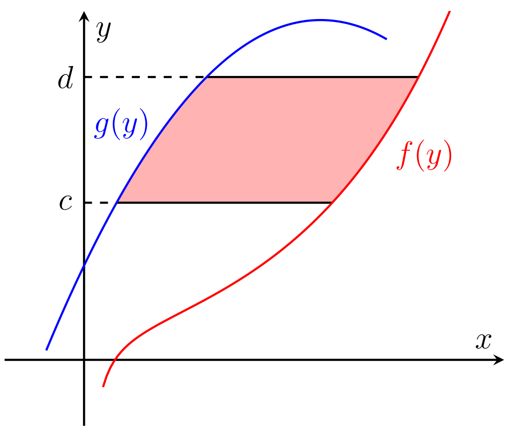
h) $\displaystyle\int_0^{\pi/2} (1+3\sin\theta)^{3/2}\cos\theta\,d\theta$

Area Between Curves: Let f and g be continuous with $f(x) \geqslant g(x)$ on $[a,b]$. Then the area of the region bounded by the curves $y=f, y=g$ and the vertical lines $x=a, x=b$ is:



$$A = \int_a^b \Big(f(x)-g(x)\Big)\,dx$$

It is also possible to integrate with respect to y. Then,



$$A = \int_c^d \Big(f(y)-g(y)\Big)\,dy$$

Note that we should integrate:
TOP minus BOTTOM, or
RIGHT minus LEFT.

Exercise 12-6: Find the area of the region bounded above by $y=x+6$, below by $y=x^2$ and bounded on the sides by the lines $x=0,\ x=2$.

Solution:

$$\begin{aligned} A &= \int_0^2 x + 6 - x^2\,dx \\ &= \frac{x^2}{2} + 6x - \frac{x^3}{3}\Big|_0^2 \\ &= 2 + 12 - \frac{8}{3} - 0 \\ &= \frac{34}{3} \end{aligned}$$

Exercise 12-7: Sketch the region enclosed by the given curves, and find its area.

a) $y = x^2,\ \ y = x + 6$ $\left(\textbf{Answer: } \frac{125}{6}\right)$

b) $x = y^2,\ \ x = y + 2$ $\left(\textbf{Answer: } \frac{9}{9}\right)$

c) $y = x^2,\ \ y = \sqrt{x},\ \ x = 1/4,\ \ x = 1$ $\left(\textbf{Answer: } \frac{49}{192}\right)$

d) $y = 2+|x-1|,\ \ y = -\frac{1}{5}x+7$ $\left(\textbf{Answer: } 24\right)$

e) $y = 4-x^2,\ \ y = 3x^2-12$ $\left(\textbf{Answer: } \frac{128}{3}\right)$

f) $y = x^2,\ \ y = a^2$ $\left(\textbf{Answer: } \frac{4a^3}{3}\right)$

Integration by Parts: We know that

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

therefore:

$$d(uv) = v\,du + u\,dv$$

$$u\,dv = d(uv) - v\,du$$

Integrating both sides, we obtain:

$$\int u\,dv = uv - \int v\,du$$

Exercise 12-8: Evaluate

$$\int xe^x\,dx$$

Solution: We have to choose u and dv correctly.

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = e^x\,dx \quad \Rightarrow \quad v = e^x$$

Using the integration by parts formula, we obtain

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$\begin{aligned} \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c \end{aligned}$$

(Check that the alternative choice $u = e^x,\ dv = xdx$ does NOT work)

Exercise 12-9: Evaluate the following integrals:

a) $\int x \sin x \, dx$

b) $\int x e^{3x} \, dx$

c) $\int x^2 e^x \, dx$

d) $\int x^2 \ln x \, dx$

e) $\int \frac{\ln x}{\sqrt{x}} \, dx$

f) $\int \ln x \, dx$

Definite Integrals using Integration by Parts:

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Exercise 12-10: Evaluate the following integrals:

a) $\int_0^2 x e^{-x} \, dx$

b) $\int_1^e \sqrt{x} \ln x \, dx$

Review Exercises

Exercise 12-11: Evaluate the following integrals:

a) $\int \sin^7 x \, \cos x \, dx$

b) $\int \sin^7 x \, \cos^3 x \, dx$

c) $\int \frac{1}{x \ln x} \, dx$

d) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

e) $\int \frac{\sin x \, \cos x}{3 + 4\sin^2 x} \, dx$

f) $\int \frac{e^{-\frac{1}{x}}}{x^2} \, dx$

g) $\int \frac{1}{3x+1} dx$

h) $\int \frac{2x+1}{x^2+x+1} dx$

Exercise 12-12: Evaluate the following definite integrals.

a) $\int_{\pi/12}^{\pi/9} \sec^2 3x \, dx$

b) $\int_e^{e^2} \frac{(\ln x)^3 \, dx}{x}$

c) $\int_0^{\pi/2} \frac{\cos x}{5 - 2\sin x} \, dx$

Exercise 12-13: Find the area enclosed by the following curves:

a) $y = 3x + 7,\ y = -x^2 + 1,\ x = -1,\ x = 4$

b) $y = x^2 + 4x + 4,\ y = 3x + 10$

c) $y = 2 - x^2,\ y = -x$

d) $y = \sqrt{x},\ y = -\sqrt{x},\ x = 4$

Exercise 12-14: Evaluate the following integrals:

a) $\displaystyle\int x^p \ln x \, dx$

b) $\displaystyle\int \arctan x \, dx$

c) $\displaystyle\int x \sec^2 x \, dx$

d) $\displaystyle\int \cos x e^x \, dx$

e) $\displaystyle\int_0^1 x^2 e^x \, dx$

— End of WEEK —

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