Week 13 – Trigonometric Integrals

Powers of Sines and Cosines

To evaluate trigonometric integrals of the type

$$\int \sin^n x \, \cos^m x \, dx$$

- If n is odd, use the substitution:

  $$u = \cos x, \quad du = -\sin x \, dx$$

  and then the identity $\cos^2 x + \sin^2 x = 1$

  $$\left(\text{if } m \text{ is odd, try:} \quad u = \sin x, \quad du = \cos x \, dx \right)$$

  If both are odd, it doesn't matter.

- If both n and m are even, use the identities

  $$\sin^2 x = \frac{1 - \cos 2x}{2}$$

  $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

  to simplify the integral.

Exercise 13-1: Evaluate $\int \sin^3 x \cos^2 x \, dx$

Solution: Using $u = \cos x$, $du = -\sin x \, dx$ we obtain:

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \, \sin x \, dx \\
&= \int (1 - u^2) u^2 \, (-du) \\
&= \int u^4 - u^2 \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + c \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c
\end{aligned}$$

Exercise 13-2: Evaluate $\int \sin^2 x \cos^2 x \, dx$

Solution:

$$\begin{aligned}
&\int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \, dx \\
&= \frac{1}{4} \int 1 - \cos^2 2x \, dx \\
&= \frac{1}{4} \int 1 - \frac{1 + \cos 4x}{2} \, dx \\
&= \frac{1}{8} \int 1 - \cos 4x \, dx \\
&= \frac{x}{8} - \frac{\sin 4x}{32} + c
\end{aligned}$$

Exercise 13-3: Evaluate $\int \cos^7 x \, dx$

Solution:

$$\begin{aligned}
\int \cos^7 x \, dx &= \int \cos^6 x \cdot \cos x \, dx \\
&= \int (1 - \sin^2 x)^3 \cdot \cos x \, dx \\
&= \int (1 - u^2)^3 \, du \qquad (u = \sin x) \\
&= \int (1 - 3u^2 + 3u^4 - u^6) \, du \\
&= u - u^3 + \frac{3}{5} u^5 - \frac{1}{7} u^7 + c
\end{aligned}$$

$$= \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$$

Exercise 13-4: Evaluate the following integrals:

a) $\int \sin^6 x \cos^3 x \, dx$

b) $\int \sin^5 x \cos^5 x \, dx$

c) $\int \sin^8 x \cos x \, dx$

d) $\int \cos^4 x \, dx$

e) $\int \sin^4 x \cos^4 x \, dx$

Products of Sines and Cosines

Adding the identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

we obtain:

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

In other words

$$\sin A \cos B = \frac{1}{2}\Big(\sin(A+B) + \sin(A-B)\Big)$$

Similarly, starting with

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

we obtain:

$$\cos A \cos B = \frac{1}{2}\Big(\cos(A+B) + \cos(A-B)\Big)$$

$$\sin A \sin B = \frac{1}{2}\Big(\cos(A-B) - \cos(A+B)\Big)$$

Exercise 13-5: Evaluate $\int \sin 7x \cos 8x \, dx$

Solution: Using the above identities, we obtain:

$$\int \sin 7x \cos 8x \, dx = \frac{1}{2} \int \big(\sin 15x - \sin x\big) \, dx$$

$$= -\frac{\cos 15x}{30} + \frac{\cos x}{2} + c$$

Exercise 13-6: Evaluate the following integrals:

a) $\int \sin 5x \sin 3x \, dx$

b) $\int \sin 4x \cos 8x \, dx$

c) $\int \cos 3x \cos \frac{x}{2} \, dx$

Integrals containing secant and tangent

Remember the formulas:

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

and

$$\sec^2 x = 1 + \tan^2 x$$

Exercise 13-7: Evaluate $\int \sec x \, dx$

Solution: Using the derivative formulas for $\sec x$ and $\tan x$, we obtain:

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \, (\sec x + \tan x)$$

$$\int \sec x \, dx = \int \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x} \, dx$$

The substitution $u = \sec x + \tan x$ gives:

$$= \int \frac{du}{u}$$

$$= \ln |u| + c$$

$$= \ln |\sec x + \tan x| + c$$

Exercise 13-8: Evaluate $\int \tan^2 x\,dx$

Solution:

$$\begin{aligned}\int \tan^2 x\,dx &= \int \left(\sec^2 x - 1\right) dx \\ &= \tan x - x + c\end{aligned}$$

Exercise 13-9: Evaluate $\int \tan^3 x\,dx$

Solution:

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan^2 x \cdot \tan x\,dx \\ &= \int (\sec^2 x - 1) \cdot \tan x\,dx \\ &= \int \sec^2 x \cdot \tan x\,dx - \int \tan x\,dx \\ &= \int u\,du - \int \tan x\,dx\end{aligned}$$

where $u = \tan x$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + c$$

Exercise 13-10: Evaluate the following integrals:

a) $\int \tan^4 x\,dx$

b) $\int \tan^3 x \sec^2 x\,dx$

c) $\int \tan^3 x \sec x\,dx$

d) $\int \tan^2 x \sec^4 x\,dx$

Trigonometric Substitutions

Using certain trigonometric identities, we can simplify integrals involving square roots:

- For $\sqrt{a^2 - x^2}$, use $x = a\sin\theta$ to obtain: $a|\cos\theta|$, $\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$
- For $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$ to obtain: $a|\sec\theta|$, $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- For $\sqrt{x^2 - a^2}$, use $x = a\sec\theta$ to obtain: $a|\tan\theta|$, $\quad 0 \leqslant \theta < \frac{\pi}{2}$ if $x \geqslant a$
  $\frac{\pi}{2} < \theta \leqslant \pi$ if $x \geqslant -a$

Exercise 13-11: Evaluate the following integrals:

a) $\int \frac{dx}{\sqrt{1 - x^2}}$

b) $\int \frac{dx}{\sqrt{9 - x^2}}$

c) $\int \sqrt{4 - x^2}\,dx$

d) $\int \frac{x\,dx}{\sqrt{4 - x^2}}$

e) $\int \frac{x\,dx}{\sqrt{4 + x^2}}$

f) $\int \frac{dx}{\sqrt{9 + x^2}}$

g) $\int \frac{dx}{\sqrt{x^2 - 16}}$

h) $\int \frac{x^2\,dx}{\sqrt{16 - x^2}}$

Review Exercises

Exercise 13-12: Evaluate the following integrals:

a) $\displaystyle\int \cos^6 x \, dx$

b) $\displaystyle\int \cos^6 x \sin^3 x \, dx$

c) $\displaystyle\int \cos(\pi x) \, \sin(2\pi x) \, dx$

d) $\displaystyle\int \cos^3(\pi x) \, dx$

e) $\displaystyle\int \tan^8 x \sec^4 x \, dx$

f) $\displaystyle\int \tan^3 x \sec^5 x \, dx$

g) $\displaystyle\int \sec^3 x \, dx$

h) $\displaystyle\int \frac{x^2 \, dx}{\sqrt{25 - x^2}}$

i) $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 16}}$

j) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 4}}$

k) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 1}}$

— End of WEEK —

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