Week 14 – Partial Fractions Improper Integrals

Partial Fractions Expansion

Given an algebraic expression like

$$\frac{3}{x-2}+\frac{5}{x+4}$$

we can write it with a common denominator as:

$$\frac{3}{x-2}+\frac{5}{x+4}=\frac{8x+2}{(x-2)(x+4)}$$

To evaluate the integrals like

$$\int \frac{(8x+2)}{(x-2)(x+4)}\,dx$$

we have to reverse this process.

$$\frac{8x+2}{(x-2)(x+4)}=\frac{A}{x-2}+\frac{B}{x+4}$$

$$8x+2=A(x+4)+B(x-2)$$

$$x=2 \quad \Rightarrow \quad A=\frac{18}{6}=3$$

$$x=-4 \quad \Rightarrow \quad B=\frac{-30}{-6}=5$$

Now, the integral can easily be evaluated in terms of logarithms.

For a given rational function $R(x)=\dfrac{P(x)}{Q(x)}$, keep in mind the following:

- If degree of $P(x)$ is greater than the degree of $Q(x)$, divide them using polynomial division.
- If $Q(x)$ contains a power of the type $(ax+b)^n$, include all the terms

  $$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$

  in the expansion.
- An irreducible second order polynomial x^2+bx+c can be transformed into $u^2 \pm a^2$ with the substitution $u=x+\dfrac{b}{2}$.

Exercise 14-1: Evaluate $\displaystyle\int \frac{4x+4}{(x-3)(x-2)}\,dx$

Solution:

$$\frac{4x+4}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}$$

$$4x+4=A(x-2)+B(x-3)$$

$$x=3 \quad \Rightarrow \quad A=16$$

$$x=2 \quad \Rightarrow \quad B=-12$$

$$\int\left(\frac{16}{x-3}-\frac{12}{x-2}\right)dx$$

$$=16\ln|x-3|-12\ln|x-2|+c$$

Exercise 14-2: Evaluate

$$\int \frac{5x^3-12x^2-6x-5}{x^2-2x-3}\,dx$$

Solution: First, we have to make a polynomial division to obtain:

$$\frac{5x^3-12x^2-6x-5}{x^2-2x-3}=5x-2+\frac{5x-11}{(x-3)(x+1)}$$

$$\frac{5x-11}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$$

$$5x-11=A(x+1)+B(x-3)$$

$$x=3 \quad \Rightarrow \quad A=1$$

$$x=-1 \quad \Rightarrow \quad B=4$$

$$\int\left(5x-2+\frac{1}{x-3}+\frac{4}{x+1}\right)dx$$

$$=\frac{5}{2}x^2-2x+\ln|x-3|+4\ln|x+1|+c$$

Exercise 14-3: Evaluate

$$\int \frac{10x^2 - 22x + 7}{x(x-1)^2}\, dx$$

Solution:

$$\frac{10x^2 - 22x + 7}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$10x^2 - 22x + 7 = A(x-1)^2 + Bx(x-1) + Cx$$

Solving these equations, we obtain:

$$A = 7, \quad B = 3, \quad C = -5$$

$$\int \left(\frac{7}{x} + \frac{3}{x-1} - \frac{5}{(x-1)^2} \right) dx$$

$$= 7 \ln|x| + 3 \ln|x-1| + \frac{5}{x-1} + c$$

Exercise 14-4: Evaluate

$$\int \frac{8x^2 + 11x + 18}{x^3 + 4x^2 + x + 4}\, dx$$

Solution:

$$\begin{aligned} \frac{8x^2 + 11x + 18}{x^3 + 4x^2 + x + 4} &= \frac{8x^2 + 11x + 18}{(x^2+1)(x+4)} \\ &= \frac{Ax + B}{x^2 + 1} + \frac{C}{x+4} \end{aligned}$$

$$8x^2 + 11x + 18 = Ax(x+4) + B(x+4) + C(x^2+1)$$

Solving these equations, we obtain:

$$A = 2, \quad B = 3, \quad C = 6$$

$$\int \left(\frac{2x+3}{x^2+1} + \frac{6}{x+4} \right) dx$$

$$= \ln|x^2+1| + 3 \arctan x + 6 \ln|x+4| + c$$

Exercise 14-5: Evaluate $\displaystyle\int \frac{dx}{x^2 + 4x + 5}$

Solution: The substitution $u = x + 2$ gives:

$$x^2 + 4x + 5 = u^2 + 1, \quad du = dx$$

$$\begin{aligned} \int \frac{1}{x^2 + 4x + 5}\, dx &= \int \frac{1}{u^2+1}\, du \\ &= \arctan u + c \\ &= \arctan(x+2) + c \end{aligned}$$

Exercise 14-6: Evaluate the following integrals:

a) $\displaystyle\int \frac{dx}{x^2 + x - 6}$

b) $\displaystyle\int \frac{6x^3 - 18x}{(x^2-1)(x^2-4)}\, dx$

c) $\displaystyle\int \frac{4x^4 + x + 1}{x^5 + x^4}\, dx$

d) $\displaystyle\int \frac{1}{4x^2 + 4x - 3}\, dx$

e) $\displaystyle\int \frac{3x+1}{(x^2 + 2x + 5)^2}\, dx$

f) $\displaystyle\int \frac{1}{\sqrt{3 - 2x - x^2}}\, dx$

Improper Integrals

The integral

$$\int_a^b f(x)\,dx$$

exists if $[a,b]$ is a closed and bounded interval and f is a continuous function. If the interval is unbounded like

$$[a,\infty), (-\infty,+\infty)$$

or if the function f has an infinite discontinuity, then the integral is called an improper integral. We define improper integrals by a limit process.

Improper Integrals of type I: Assuming the function f is continuous on the region of integration:

- $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
- $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$
- $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^{+\infty} f(x)\,dx$

If those limits exist, we say the integral converges, otherwise, we say the integral diverges.

Exercise 14-7: Evaluate $\int_1^\infty \frac{dx}{x^2}$

Solution:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2} &= \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} \\ &= \lim_{t\to\infty} -\frac{1}{x}\bigg|_1^t \\ &= \lim_{t\to\infty} 1-\frac{1}{t} \\ &= 1\end{aligned}$$

So the integral is convergent.

Exercise 14-8: Evaluate $\int_1^\infty \frac{dx}{\sqrt{x}}$

Solution:

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}} &= \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}} \\ &= \lim_{t\to\infty} 2\sqrt{x}\bigg|_1^t \\ &= \lim_{t\to\infty} 2\sqrt{t}-2 \\ &= \infty\end{aligned}$$

So the integral is divergent.

Improper Integrals of type II:

- If f is continuous on $(a,b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$
- If f is continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$
- If f is continuous on $[a,b]$ except for c where $a<c<b$ then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example: Evaluate $\int_0^2 \frac{dx}{(x-1)^2}$

Solution:

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \lim_{t\to1^-} \int_0^t \frac{dx}{(x-1)^2} + \lim_{t\to1^+} \int_t^2 \frac{dx}{(x-1)^2}$$

$$\begin{aligned}\lim_{t\to1^-} \int_0^t \frac{dx}{(x-1)^2} &= \lim_{t\to1^-} -\frac{1}{x-1}\bigg|_0^t \\ &= \lim_{t\to1^-} -\frac{1}{t-1}-1 \\ &= \infty\end{aligned}$$

So the integral is divergent. Note that

$$\int_0^2 \frac{dx}{(x-1)^2} = -\left.\frac{1}{x-1}\right|_0^2$$

gives an incorrect result.

Direct Comparison Test: Suppose that f and g are continuous on $[a,\infty)$ and

$$0 \leqslant f(x) \leqslant g(x)$$

for all $x \in [a,\infty)$.

- If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.
- If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

Exercise 14-9: Is the integral $\displaystyle\int_1^\infty \frac{dx}{x^2+x^5}$ convergent?

Solution: It is difficult to evaluate this integral, but we don't have to. We know that

$$x > 1 \quad\Rightarrow\quad x^2 + x^5 > x^2$$
$$\Rightarrow\quad \frac{1}{x^2+x^5} < \frac{1}{x^2}$$

$\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent, therefore $\displaystyle\int_1^\infty \frac{dx}{x^2+x^5}$ is also convergent by comparison test.

Limit Comparison Test: Suppose that positive functions f and g are continuous on $[a,\infty)$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where $0 < L < \infty$, then the integrals

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Exercise 14-10: Is the integral $\displaystyle\int_1^\infty \frac{5+2x}{3+x^2}\,dx$ convergent?

Solution: It is difficult to evaluate this integral, but we don't have to. We know that

$$\lim_{x\to\infty} \frac{\dfrac{5+2x}{3+x^2}}{\dfrac{1}{x}} = 2$$

$\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent, therefore $\displaystyle\int_1^\infty \frac{5+2x}{3+x^2}\,dx$ is also divergent by limit comparison test.

Review Exercises

Exercise 14-11: Evaluate the following integrals:

a) $\displaystyle\int \frac{x^2}{x^4-1}\,dx$

b) $\displaystyle\int \frac{x^4}{x^2+4}\,dx$

c) $\displaystyle\int \frac{dx}{x^3+x}$

d) $\displaystyle\int \frac{dx}{\sqrt{5+12x-9x^2}}$

e) $\displaystyle\int x\sqrt{3+2x-x^2}\,dx$

Exercise 14-12: Evaluate the following integrals if they converge:

a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

c) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

d) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$

e) $\displaystyle\int_3^4 \frac{dx}{(x-3)^2}$

f) $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx$

Exercise 14-13: Determine the convergence or divergence of the following integrals:

a) $\displaystyle\int_0^\infty \frac{1}{x+e^x}\,dx$

b) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

c) $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx$

d) $\displaystyle\int_1^\infty \frac{x^2e^x}{\ln x}\,dx$

e) $\displaystyle\int_\pi^\infty \frac{dx}{\sqrt{x}-\sin x}$

Answers

Exercise 14-11:

a) $\displaystyle\frac{1}{2}\arctan x+\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+c$

b) $\displaystyle\frac{x^3}{3}-4x+8\arctan\frac{x}{2}+c$

c) $\displaystyle\ln|x|-\frac{1}{2}\ln|x^2+1|+c$

d) $\displaystyle\frac{1}{3}\arcsin\left(x-\frac{2}{3}\right)+c$

e) $\displaystyle\frac{1}{6}(2x^2-x-9)\sqrt{3+2x-x^2}$
$\displaystyle+2\arcsin\frac{x-1}{2}+c$

Exercise 14-12:

a) 1

b) 2

c) $\displaystyle\frac{1}{p-1}$ if $p>1$

d) $3+3\sqrt[3]{2}$

e) Divergent

f) Divergent

Exercise 14-13:

a) Convergent

b) Convergent

c) Divergent

d) Divergent

e) Divergent

— End of WEEK —

Author: Dr. Emre Sermutlu

Last Update: December 22, 2016