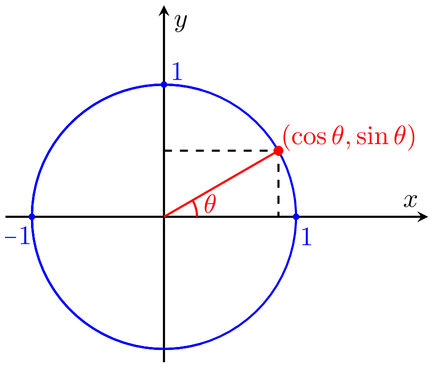
Week 2 – Precalculus II

Trigonometric Functions

The functions sine and cosine are defined on the unit circle as

$$\cos\theta = x,\ \sin\theta = y$$



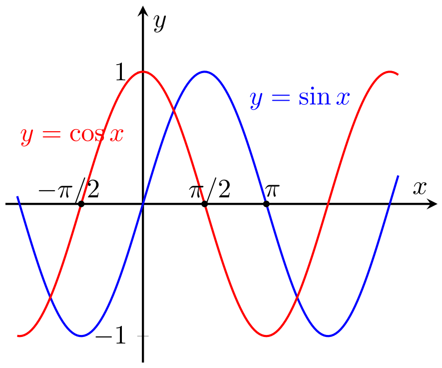
We will always use radian measure for the trigonometric functions.

$$\frac{\text{radian}}{\pi} = \frac{\text{degree}}{180}$$

The functions $\sin\theta$ and $\cos\theta$ are periodic with period 2π, so:

$$\sin(\theta + 2n\pi) = \sin\theta$$

$$\cos(\theta + 2n\pi) = \cos\theta$$



The function $\tan\theta$ is defined as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The other important trigonometric functions are:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

We can find the values of trigonometric functions for $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ etc. using special triangles.

Some important trigonometric formulas are:

$$\cos(-\theta) = \cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

Exercise 2-1: Using $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, calculate $\sin\frac{\pi}{8}$.

Exercise 2-2: Express $\tan(x+y)$ in terms of $\tan x$ and $\tan y$.

Exercise 2-3: Evaluate the following:

a) $\tan\frac{3\pi}{4}$

b) $\sin\frac{2\pi}{2}$

c) $\cos\frac{5\pi}{6}$

d) $\sin\left(x+\frac{\pi}{2}\right)$

Exercise 2-4: Prove the following formulas:

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

Exercise 2-5: Find the domains of the following functions:

a) $f(x) = \tan x$

b) $f(x) = \tan\left(\frac{x}{3}+\frac{\pi}{2}\right)$

c) $f(x) = 12\sin\left(\frac{x^2-3}{4}\right)$

Exercise 2-6: Sketch the graph of the following functions:

a) $f(x) = \sin^2 x$

b) $f(x) = \cos(2\pi x)$

c) $f(x) = \sin\left(x-\frac{\pi}{2}\right)$

d) $f(x) = 2\sin x + 3$

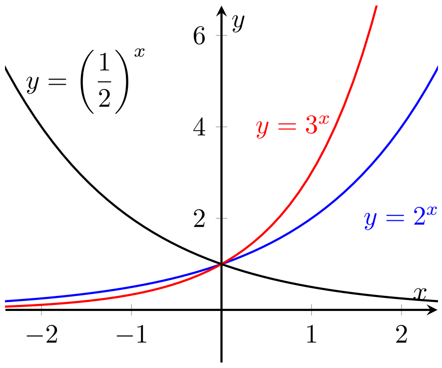
Exponential Functions

Functions of the form

$$f(x) = a^x$$

where a is a positive constant (but $a \neq 1$) are called exponential functions.

The domain of an exponential function is: $\mathbb{R} = (-\infty, \infty)$ and the range is: $(0, \infty)$.



Note that:

- $a^n = a \cdot a \cdots a$
- $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$

- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Exercise 2-7: If we invest an amount A in the bank, and if the rate of interest is 15% per year, how much money will we have after n years?

Exercise 2-8: The price of a house doubles every 5 years. If the price is P now, what will be the price after n years?

Exercise 2-9: A firm has C customers. Every month, 30% of the customers leave. How many remain after n months?

Rules for Exponents:

- $a^x \cdot a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $a^x \cdot b^x = (ab)^x$

The natural exponential function is:

$$f(x) = e^x$$

where $e = 2.71828\ldots$. This exponential has many simple properties.

Inverse Functions:

If $\quad f(g(x)) = x \quad$ and $\quad g(f(x)) = x$

the functions f and g are inverses of each other. For example, the inverse of $f(x) = 2x + 1$ is $g(x) = \dfrac{x-1}{2}$

Question: Does each function have an inverse?

One–to–one Functions: If

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

then f is one-to one.

Onto Functions: Let $f : A \to B$. If there exists an $x \in A$ for all $y \in B$ such that $f(x) = y$ then f is onto.

Theorem: A function has an inverse if and only if it is one-to-one and onto.

Exercise 2-10: Find the inverse of the following functions. (If possible)

a) $y = x^3$

b) $y = x^4$

c) $y = \sin x$

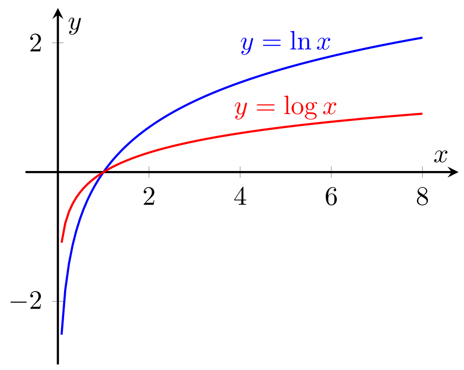
Logarithmic Functions

The inverse of the exponential function $y = a^x$ is the logarithmic function with base a:

$$y = \log_a x$$

where $a > 0, \quad a \neq 1$, in other words

$$a^{\log_a x} = \log_a(a^x) = x$$



We will use the following shorthand notations:

$\log x$	for	$\log_{10} x$	(common logarithm)
$\ln x$	for	$\log_e x$	(natural logarithm)

We can easily see that,

$$a^x \cdot a^y = a^{x+y} \quad \Rightarrow \quad \log_a(AB) = \log_a A + \log_a B$$

As a result of this,

- $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$
- $\log_a \left(\frac{1}{B}\right) = -\log_a B$
- $\log_a (A^r) = r \log_a A$

Exercise 2-11: Simplify the following expressions as much as possible:

a) $\log 1000$

b) $\ln 72$

c) $\log 50000$

d) $\log_3 \sqrt{3}$

e) $\log_2 32$

f) $\log_4 8$

g) $\log_5 \frac{1}{125}$

h) $\log 0.0012$

Exercise 2-12: What is the domain of the following functions?

a) $f(x) = \ln(\ln x)$

b) $f(x) = \ln(\ln(\ln x))$

Any logarithm can be expressed in terms of the natural logarithm:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Any exponential can be expressed in terms of the natural exponential:

$$a^x = e^{x \ln a}$$

Review Exercises

Exercise 2-13: Given that $\sin x = \dfrac{1}{3}$, find $\cos x$ and $\tan x$.

Exercise 2-14: Given that $\sin x = -\dfrac{1}{2}$, find x.

Exercise 2-15: Given that $\cos\theta = 0.9$, find $\cos\dfrac{\theta}{2}$ and $\sin\dfrac{\theta}{2}$.

Exercise 2-16: Find $\tan\dfrac{8\pi}{3}$.

Exercise 2-17: Find $\sec\dfrac{-3\pi}{4}$.

Exercise 2-18: Sketch the graph of

$$f(x) = 4 + 5\cos^2 x$$

Exercise 2-19: Find the domain and range of the following functions:

a) $f(x) = 3 - e^{-x^2}$

b) $f(x) = |-5 + \sin^2 x|$

c) $f(x) = \tan x$

d) $f(x) = \tan^2(x - \pi)$

e) $f(x) = \ln(\ln x)$

f) $f(x) = \ln(\ln(\ln x))$

Exercise 2-20: Sketch the following functions:

a) $y = e^{|x|}$

b) $y = e^{-|x|}$

c) $y = \ln(3x - 1)$

d) $y = \ln\dfrac{1}{x}$

e) $y = e^{-2\ln x}$

f) $y = \left(\dfrac{1}{2}\right)^x$

Exercise 2-21: Simplify the following expressions as much as possible:

a) $2^{\log_2 5}$

b) $2^{\log_4 5}$

c) $e^{1+\ln 17}$

d) $e^{\ln 7 - \ln(12)}$

e) $10^{3/2 - \log(300)}$

— End of WEEK —

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