Week 3 – Limits

Informal Definition of Limit: We say that $f(x)$ has the limit L at $x = a$ if $f(x)$ gets as close to L as we like, when x approaches a. (without getting equal to a) We write this as:

$$\lim_{x \to a} f(x) = L$$

Exercise 3-1: Investigate the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution:

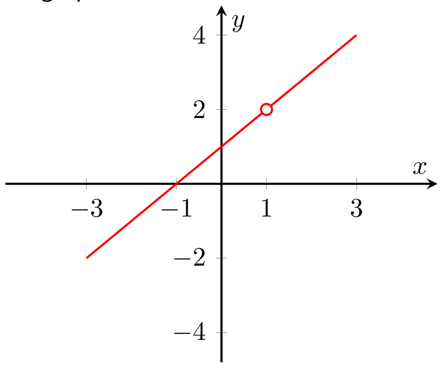
x	f
0.9	1.9
0.99	1.99
0.999	1.999
⋮	⋮

x	f
1.1	2.1
1.01	2.01
1.001	2.001
⋮	⋮

These results suggest that the limit is 2.
Actually, the function can be written as:

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \neq 1 \\ \text{undefined} & \text{if} \quad x = 1 \end{cases}$$

Its graph is:



Exercise 3-2: Investigate what happens to

$$f(x) = \frac{\sin x}{x}$$

as $x \to 0$ using a calculator.

Solution:

x	f
0.1	0.998334
0.05	0.999583
0.01	0.999983
0.005	0.999996
⋮	⋮

This suggests that the limit is 1.

Exercise 3-3: Evaluate the following limits:

a) $\lim\limits_{x \to 1} \dfrac{x^2 - 9}{x - 3}$

b) $\lim\limits_{x \to 0} \dfrac{|x|}{x}$

c) $\lim\limits_{x \to 2} \dfrac{x - 1}{x + 1}$

d) $\lim\limits_{x \to 1} \dfrac{1}{x^2 - 1}$

e) $\lim\limits_{x \to 2} \dfrac{x^2 - 7x + 10}{x^2 - 5x + 6}$

f) $\lim\limits_{x \to 4} \dfrac{x^2 - 7x + 10}{x^2 - 5x + 6}$

g) $\lim\limits_{x \to 6} \dfrac{x^2 - 5x + 4}{x - 6}$

Limit Laws: If both of the limits

$$\lim_{x \to a} f(x) = L$$

and

$$\lim_{x \to a} g(x) = M$$

exist, then:

- $\lim\limits_{x \to a} f \pm g = L \pm M$
- $\lim\limits_{x \to a} fg = LM$
- $\lim\limits_{x \to a} \dfrac{f}{g} = \dfrac{L}{M}$ (if $M \neq 0$)
- $\lim\limits_{x \to a} \sqrt[n]{f} = \sqrt[n]{L}$
- $\lim\limits_{x \to a} f(g(x)) = f(M)$
  (If f is continuous at M)

Exercise 3-4: Evaluate the following limits:

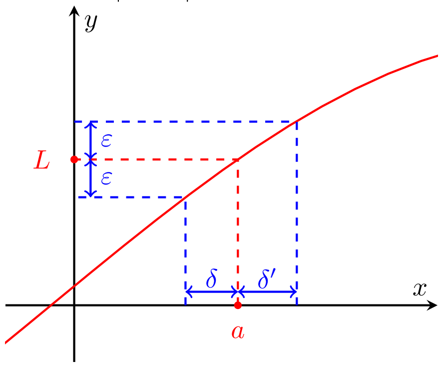
a) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^4 - 1}$

b) $\lim\limits_{x \to 4} \dfrac{2 - \sqrt{x}}{4 - x}$

c) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$

d) $\lim\limits_{x \to -2} \dfrac{(x + 2)^2}{x^4 - 16}$

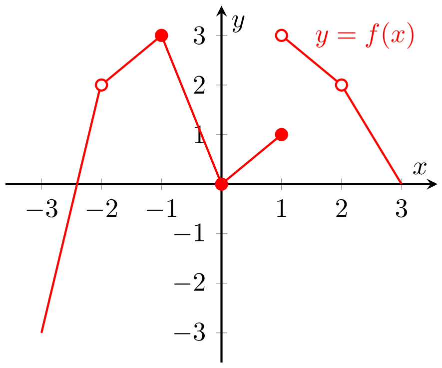
Formal Definition of Limit: The number L is the limit of $f(x)$ as x approaches a if given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$



Exercise 3-5: Using the $\varepsilon - \delta$ definition, show that $\lim\limits_{x \to 1} 3x + 2 = 5$.

Answer: $\delta = \dfrac{\varepsilon}{3}$

Exercise 3-6: Find the following limits for $f(x)$ based on the figure: (if they exist)



a) $\lim_{x \to -2} f(x)$

b) $\lim_{x \to -1} f(x)$

c) $\lim_{x \to 0} f(x)$

d) $\lim_{x \to 1} f(x)$

e) $\lim_{x \to 2} f(x)$

Sandwich (Squeeze) Law: If

$$f(x) \leqslant g(x) \leqslant h(x)$$

and if

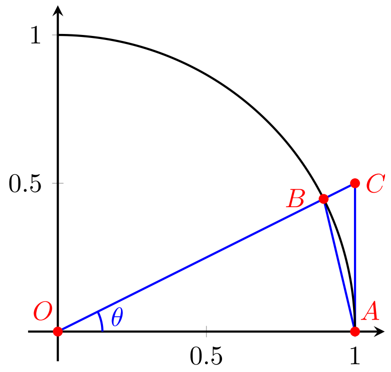
$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then

$$\lim_{x \to a} g(x) = L$$

Exercise 3-7: Show that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ using Sandwich Law.

Solution: Consider the triangles and sectors on the unit circle:



Area of $\Delta AOB = \dfrac{\sin\theta \cdot 1}{2}$

Area of $\Delta AOC = \dfrac{\tan\theta \cdot 1}{2}$

If you remember the radian definition of θ,

Area of circular sector $AOB = \dfrac{1 \cdot 1 \cdot \theta}{2}$

From the figure we can see that:

area(ΔAOB) < area(sectorAOB) < area(ΔAOC)

$$\frac{1}{2}\sin\theta < \frac{1}{2}\,\theta < \frac{1}{2}\tan\theta$$

$$\Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow \quad \cos\theta < \frac{\sin\theta}{\theta} < 1$$

Now, using the sandwich theorem gives the result we want, because $\lim_{\theta\to 0}\cos\theta = 1$

Exercise 3-8: Evaluate the following limits:

a) $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$

b) $\lim_{x\to 0}\frac{\sin 7x}{2x}$

c) $\lim_{x\to 0}\frac{\tan x}{2x}$

d) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

e) $\lim_{x\to 0}\frac{x^2\tan x}{\sin^3\sqrt{x}}$

f) $\lim_{x\to 0}\frac{x^3}{\sin^2(5x)}$

Exercise 3-9: Do the following limits exist?

a) $\lim_{x\to 0}\sin\frac{1}{x}$

b) $\lim_{x\to 0}x\sin\frac{1}{x}$

Exercise 3-10: Evaluate the limit:

$$\lim_{x\to\pi}\frac{\sin^2 x}{1+\cos^3 x}$$

Solution:

$$\sin^2 x = 1-\cos^2 x = (1-\cos x)(1+\cos x)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

Therefore

$$1+\cos^3 x = (1+\cos x)(1-\cos x+\cos^2 x)$$

Inserting these in the given expression, we obtain:

$$\lim_{x\to\pi}\frac{\sin^2 x}{1+\cos^3 x}$$
$$= \lim_{x\to\pi}\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)(1-\cos x+\cos^2 x)}$$
$$= \lim_{x\to\pi}\frac{1-\cos x}{1-\cos x+\cos^2 x}$$
$$= \frac{2}{3}$$

Exercise 3-11: Evaluate the limit:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$$

(Hint: $\frac{1-\cos x}{\cos x\sin^2 x} = \frac{1}{\cos x(1+\cos x)} = \frac{1}{2}$)

Review Exercises

Exercise 3-12: Evaluate the following limits: (If they exist)

a) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x^2-7x+12}$

b) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{(x-3)^2}$

c) $\displaystyle\lim_{x\to 3} \frac{(x-3)^2}{x^2-9}$

d) $\displaystyle\lim_{x\to 0} \frac{x^4-5x^2+12x+7}{5x^2+6}$

e) $\displaystyle\lim_{x\to 1} \frac{2}{1-x^4}-\frac{1}{1-x^2}$

Exercise 3-13: Evaluate the following limits: (If they exist)

a) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-25}$

b) $\displaystyle\lim_{x\to 1} \frac{1-\sqrt{x}}{1-\sqrt[3]{x}}$

c) $\displaystyle\lim_{x\to 0} \frac{x}{\sqrt{a+bx}-\sqrt{a-cx}}$

Exercise 3-14: Evaluate the following limits: (If they exist)

a) $\displaystyle\lim_{x\to 0} \frac{\sin 2x-\sin 5x}{8x}$

b) $\displaystyle\lim_{x\to 0} \frac{\sin 2x+\tan 6x}{x}$

c) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

d) $\displaystyle\lim_{x\to \pi/2} \frac{\cos x}{x-\pi/2}$

e) $\displaystyle\lim_{x\to \pi/4} \frac{\cos x-\sin x}{\cos 2x}$

— End of WEEK —

Author: Dr. Emre Sermutlu

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