Week 4 – Continuity

One Sided Limits: If x approaches a from right, taking values larger than a only, we denote this by $x \to a^+$ If $f(x)$ approaches L as $x \to a^+$, then we say that L is the right-hand limit of f at a.

$$\lim_{x \to a^+} f(x) = L$$

We define the left-hand limit of f at a similarly:

$$\lim_{x \to a^-} f(x) = L$$

Theorem: The limit

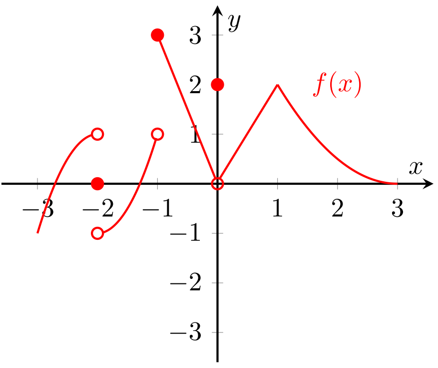
$$\lim_{x \to a} f(x) = L$$

exists if and only if both one sided limits

$$\lim_{x \to a^+} f(x) \quad \text{and} \quad \lim_{x \to a^-} f(x)$$

exist and are equal to L.

Exercise 4-1: Find the limits based on the figure:



a) $\lim_{x \to -2^-} f(x)$

b) $\lim_{x \to -2^+} f(x)$

c) $\lim_{x \to -2} f(x)$

d) $\lim_{x \to -1^-} f(x)$

e) $\lim_{x \to -1^+} f(x)$

f) $\lim_{x \to -1} f(x)$

g) $\lim_{x \to 0^-} f(x)$

h) $\lim_{x \to 0^+} f(x)$

i) $\lim_{x \to 0} f(x)$

j) $\lim_{x \to 1^-} f(x)$

k) $\lim_{x \to 1^+} f(x)$

l) $\lim_{x \to 1} f(x)$

Exercise 4-2: Find the limits $\lim_{x \to b^+} f(x)$ and $\lim_{x \to b^-} f(x)$ and graph the function:

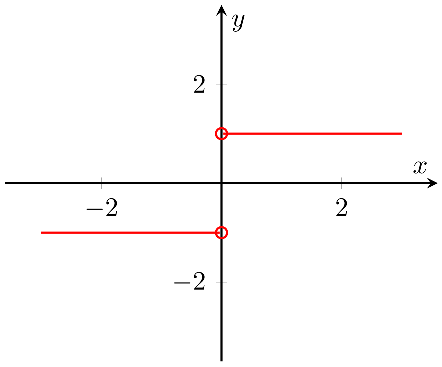
$$f(x) = \frac{x-b}{|x-b|}$$

Solution: As $x \to b^+$, $x - b > 0$ therefore $|x - b| = x - b$ and

$$\lim_{x \to b^+} f(x) = \lim_{x \to b^+} \frac{x-b}{x-b} = 1$$

Similarly,

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^-} \frac{x-b}{-(x-b)} = -1$$



Exercise 4-3: Let $f(x) = \begin{cases} 2x+1 & \text{if} \quad x < 1 \\ 5x-2 & \text{if} \quad x > 1 \end{cases}$

Find:

a) $\lim_{x \to 1^-} f(x)$

b) $\lim_{x \to 1^+} f(x)$

c) $\lim_{x \to 1} f(x)$

Exercise 4-4: Let $f(x) = \begin{cases} -x^2+2 & \text{if} \quad x < 0 \\ 7 & \text{if} \quad x = 0 \\ e^{-x} & \text{if} \quad x > 0 \end{cases}$

Find:

a) $\lim_{x \to 0^-} f(x)$

b) $\lim_{x \to 0^+} f(x)$

c) $\lim_{x \to 0} f(x)$

Exercise 4-5: Let $f(x) = \begin{cases} 4-\cos x & \text{if} \quad x < \pi \\ 0 & \text{if} \quad x = \pi \\ 5\sin\frac{x}{2} & \text{if} \quad x > \pi \end{cases}$

Find:

a) $\lim_{x \to \pi^-} f(x)$

b) $\lim_{x \to \pi^+} f(x)$

c) $\lim_{x \to \pi} f(x)$

Exercise 4-6: Find the following one-sided limits:

a) $\lim\limits_{x \to 3^+} \sqrt{\dfrac{x-3}{x+3}}$

b) $\lim\limits_{h \to 0^+} \dfrac{\sqrt{16+3h}-4}{h}$

c) $\lim\limits_{x \to -2^+} \dfrac{|x^2-4|}{x+2}$

d) $\lim\limits_{x \to -2^-} \dfrac{|x^2-4|}{x+2}$

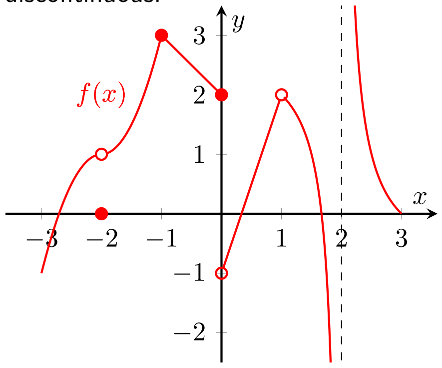
Continuity: We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

In other words:

- f must be defined at a
- $\lim\limits_{x \to a} f(x)$ must exist
- The limit must be equal to the function value.

Exercise 4-7: Determine the points where $f(x)$ is discontinuous:



Exercise 4-8: Is $f(x) = \begin{cases} \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$ continuous at $x = 0$?

Properties of Continuous Functions:

- A polynomial function is continuous on $\mathbb{R}$.
- A rational function is continuous wherever it is defined.
- The functions $\sin x$, $\cos x$ and e^x are continuous on $\mathbb{R}$.
- Any sum or product or composition of continuous functions is also continuous.

Question: Let f be an invertible function. If f is continuous, is f^{-1} always continuous?

Exercise 4-9: For what values of x is the function

$$f(x) = \ln\left(\frac{x-1}{x+2}\right)$$

continuous?

Solution: The rational function $\dfrac{x-1}{x+2}$ is continuous for all values except $x = -2$. The function $\ln x$ is continuous if $x > 0$. Therefore the answer is:

$$(-\infty, -2) \cup (1, \infty)$$

Exercise 4-10: Let $f(x) = \begin{cases} cx^2 - 2 & \text{if} \quad x \leqslant 2 \\ \dfrac{x}{c} & \text{if} \quad 2 < x \end{cases}$

Find the value c that makes f continuous.

Removable Discontinuity: Let f have a limit at $x = a$ and be discontinuous at $x = a$. Then we can make it continuous by redefining at $x = a$ only, and it is called a removable discontinuity.

Exercise 4-11: Find the points where the given function is discontinuous. Is the discontinuity removable?

a) $f(x) = \dfrac{x-5}{x^2-25}$

b) $f(x) = \dfrac{1}{1-|x|}$

c) $f(x) = \begin{cases} -1 + x & \text{if} \quad x \leqslant 0 \\ 1 + x^2 & \text{if} \quad 0 < x \end{cases}$

Exercise 4-12: Find two functions, f and g such that they are both discontinuous at $x = a$, but the discontinuity is removable for f and non-removable for g.

Exercise 4-13: Find the value of c such that f is continuous

a) $f(x) = \begin{cases} c^2 - x^2 & \text{if} \quad x \leqslant 0 \\ 2(x-c)^2 & \text{if} \quad 0 < x \end{cases}$

b) $f(x) = \begin{cases} x + c & \text{if} \quad x \leqslant \pi/2 \\ c\cos x & \text{if} \quad \pi/2 < x \end{cases}$

Continuous Extension: If a function f is undefined at $x = a$ but if $\lim\limits_{x\to a} f(x) = L$ exists, we can define a new function f_2 such that it will be continuous at $x = a$. This is called the continuous extension of f.

$$f_2(x) = \begin{cases} f(x) & \text{if} \quad x \neq a \\ L & \text{if} \quad x = a \end{cases}$$

Exercise 4-14: Find the continuous extensions of the following functions: (if possible)

a) $f(x) = \dfrac{\sin x}{x}$

b) $f(x) = \dfrac{x+6}{x^2+8x+12}$

c) $f(x) = x\sin\left(\dfrac{1}{x}\right)$

d) $f(x) = \sin\left(\dfrac{1}{x}\right)$

Theorem: Let f be a continuous function on $[a, b]$. Then for all $f(a) < x < f(b)$ there exists c in (a, b) such that $f(c) = x$.

Intermediate Value Theorem: A function $f(x)$ that is continuous on the closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Exercise 4-15: Show that there is a root of the equation $x^3 - 2x - 1 = 0$ on the interval $[0, 2]$.

Exercise 4-16: Show that there is a root of the equation $e^{-x} = x$ on the interval $[0, 1]$.

Question: Consider the following equations:

$$ax^2 + bx + c = 0$$

$$ax^3 + bx^2 + cx + d = 0$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

where $a, b, \ldots, e$ are arbitrary, but $a \neq 0$.

a) At most how many roots do they have?

b) At least how many roots do they have?

(We are only considering real roots here.)

Review Exercises

Exercise 4-17: Evaluate the following limits: (If they exist)

a) $\lim_{x\to 2^-} \sqrt{8-x^3}$

b) $\lim_{x\to 2^+} \sqrt{8-x^3}$

c) $\lim_{x\to 0^-} \ln x$

d) $\lim_{x\to 0^+} \ln x$

Exercise 4-18: Find all the discontinuities of the following functions and classify them as removable or non-removable:

a) $f(x) = \dfrac{x^2-2}{x^2-4}$

b) $f(x) = \dfrac{|x-a|}{x-a}$

c) $f(x) = \dfrac{x^2-5x+6}{x^2-4x+3}$

d) $f(x) = \tan x$

Exercise 4-19: Find the values of constants that will make the following functions continuous everywhere:

a) $f(x) = \begin{cases} a+bx^2 & \text{if} \quad x<0 \\ b & \text{if} \quad x=0 \\ 2+e^{-x} & \text{if} \quad x>0 \end{cases}$

b) $f(x) = \begin{cases} a\sin x & \text{if} \quad x<-\pi/2 \\ 1-x^2 & \text{if} \quad -\pi/2 \leqslant x \leqslant \pi/2 \\ b+\sin x & \text{if} \quad x>\pi/2 \end{cases}$

Exercise 4-20: For what values of x is the function

$$f(x) = \sqrt{3x+x^2}$$

continuous?

Exercise 4-21: Show that the equation

$$x^3-5x+1=0$$

has three different roots in the interval $[-3,3]$

— End of WEEK —

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