Week 5 – Limits Involving Infinity

- Infinity (∞) is not a number.
- $x \to \infty$ means x increases without any bounds.

Variable (x) Approaches Infinity: If the function f approaches the number L as x approaches infinity, (or negative infinity), we write:

$$\lim_{x\to\infty} f(x) = L$$

$$\left(\text{or} \quad \lim_{x\to-\infty} f(x) = L\right)$$

The formal definition is: We say f approaches L as x approaches ∞ if, given any $\varepsilon > 0$ there exists a number M such that:

$$\text{if} \quad x > M \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Exercise 5-1: Find the following limits:
(Hint: Consider the dominant terms.)

a) $\displaystyle\lim_{x\to\infty} \frac{1}{x}$

b) $\displaystyle\lim_{x\to\infty} \frac{x}{1+x^2}$

c) $\displaystyle\lim_{x\to-\infty} \frac{7x^2-12}{4x^2+18x-54}$

d) $\displaystyle\lim_{x\to\infty} \frac{x(3\sqrt{x}-x)}{10x+9}$

Exercise 5-2: Find the following limits:

a) $\displaystyle\lim_{x\to\infty} \left(\frac{2x^2-8}{3x^2-27}\right)^4$

b) $\displaystyle\lim_{x\to-\infty} \frac{x^{1/3}-5x^{7/2}}{4x^2-\sqrt{x}}$

c) $\displaystyle\lim_{x\to\infty} \frac{x^{-1}+1}{4x^{-2}+2x}$

d) $\displaystyle\lim_{x\to\infty} \frac{e^x-e^{-x}}{e^{2x}+3}$

e) $\displaystyle\lim_{x\to-\infty} \frac{e^x-e^{-x}}{e^{2x}+3}$

Exercise 5-3: Find the following limits:

a) $\displaystyle\lim_{x\to\infty} \sqrt{x+8}-\sqrt{x+4}$

b) $\displaystyle\lim_{x\to\infty} \sqrt{4x-2}-2x$

c) $\displaystyle\lim_{x\to-\infty} \sqrt{2x^2+10x}-\sqrt{2x^2-2x}$

d) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+6x-9}-\sqrt{x^2-6x-9}$

e) $\displaystyle\lim_{x\to\infty} \sqrt{3x^2-7x}-\sqrt{2x^2+32x}$

Horizontal Asymptotes: If f approaches b as
x approaches ∞ or $-\infty$, then the line

$$y = b$$

is a horizontal asymptote of the graph of $y = f(x)$.

Exercise 5-4: Find the horizontal asymptotes of the following functions: (if there's any)

a) $f(x) = \dfrac{12x+8}{x-76}$

b) $f(x) = \dfrac{2x-5}{x^2-4}$

c) $f(x) = \dfrac{4x^3+17x}{3|x|^3-8x^2+3x+19}$

d) $f(x) = e^x$

e) $f(x) = e^{-x}$

f) $f(x) = e^{1/x}$

g) $f(x) = x \sin x$

h) $f(x) = \dfrac{x - \sin x}{3x + \cos x}$

i) $f(x) = \ln x$

j) $f(x) = \sqrt{x}$

Oblique Asymptotes: If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote.

Basically, the function f behaves like $ax + b$ as $x \to \pm\infty$

Exercise 5-5: Find the oblique asymptotes of the following functions: (if there's any)

a) $f(x) = \dfrac{2x - 1}{x}$

b) $f(x) = \dfrac{x^2}{x - 1}$

c) $f(x) = \dfrac{x^4 - 3x^2}{x^2 + 1}$

d) $f(x) = \dfrac{x^3 + 3x^2 - 4x + 5}{2x^2 + x - 1}$

Solution:

Function (f) Approaches Infinity: If the value of f increases without any bound as $x \to a^+$
(or $x \to a^-$ or $x \to a$) then we say that

$$\lim_{x \to a^+} f(x) = \infty$$

This does not mean limit exists, infinity is not a number, and limit is equal to infinity is a way of saying it does not exist.

The formal definition is: We say f approaches infinity as x approaches a if, given any $M > 0$ there exists a number $\delta > 0$ such that:

$$\text{if} \quad |x - a| < \delta \quad \text{then} \quad f(x) > M$$

Exercise 5-6: Find the following limits:

a) $\displaystyle\lim_{x \to 3^+} \frac{1}{x - 3}$

b) $\displaystyle\lim_{x \to 3^-} \frac{1}{x - 3}$

c) $\displaystyle\lim_{x \to 2} \frac{(x - 2)^2}{x^2 - 4}$

d) $\displaystyle\lim_{x \to 2} \frac{(x - 2)}{x^2 - 4}$

e) $\displaystyle\lim_{x \to 2^+} \frac{(x - 4)}{x^2 - 4}$

f) $\displaystyle\lim_{x \to 2^-} \frac{(x - 4)}{x^2 - 4}$

Exercise 5-7: Find the following limits:

a) $\displaystyle\lim_{x \to 0^-} \frac{1}{1 - e^x}$

b) $\displaystyle\lim_{x \to 0^+} \frac{1}{1 - e^x}$

c) $\displaystyle\lim_{x \to 0^+} \ln x$

d) $\displaystyle\lim_{x \to \frac{3\pi}{2}^+} \tan x$

e) $\displaystyle\lim_{x \to \frac{3\pi}{2}^-} \tan x$

Vertical Asymptotes: A line $x = a$ is a vertical asymptote of the graph of the function $f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Exercise 5-8: Find the vertical asymptotes of the following functions: (if there's any)

a) $f(x) = \dfrac{x-4}{x+5}$

b) $f(x) = \dfrac{2x-5}{x^2-9}$

c) $f(x) = \tan x$

d) $f(x) = \dfrac{x}{(x+2)^2}$

e) $f(x) = \dfrac{3x-2}{5x-8}$

Review Exercises

Exercise 5-9: Find all horizontal, vertical and oblique asymptotes of the following functions: (if they exist)

a) $f(x) = \dfrac{1}{(x+9)^2}$

b) $f(x) = \dfrac{x^2}{(x+9)^2}$

c) $f(x) = \dfrac{x^3}{(x+9)^2}$

d) $f(x) = 3 - 2e^{-x}$

— End of WEEK —

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