Week 6 – Derivative

The slope of the curve $y = f(x)$ at the point x_0 is:

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists. The tangent line through the point $(x_0, f(x_0))$ has this slope.

Derivative: The derivative of the function $f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

We can think of the derivative as the rate of change of a function f or the slope of the curve of $y = f(x)$. We will use

$$y', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx} f(x)$$

to denote derivatives and

$$f'(a), \quad \left.\frac{dy}{dx}\right|_{x=a}$$

to denote their values at a certain point. Note that derivative is a function, its value at a point is a number.

Example: Instantaneous velocity is defined as

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

In mechanics, velocity is the derivative of distance and acceleration is the derivative of velocity.

Exercise 6-1: Find the derivative of $f(x) = x^2$ using the definition.

Solution: $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

Exercise 6-2: Find the derivative of $f(x) = \sqrt{x}$ using the definition.

Solution: $f'(x) = \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$$
\begin{aligned}
&= \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Exercise 6-3: Find the derivative of $f(x) = \dfrac{1}{x}$ using the definition.

Differentiation Formulas: Using the definition of derivative, we obtain:

- Derivative of a constant is zero, i.e. $\dfrac{dc}{dx} = 0$
- Derivative of $f(x) = x$ is

  $$f'(x) = 1$$

- Derivative of $f(x) = x^2$ is

  $$f'(x) = 2x$$

- Derivative of $f(x) = x^n$ (where n is a positive integer) is:

  $$f'(x) = nx^{n-1}$$

- Derivative of $f(x) = \sqrt{x}$ is:

  $$f'(x) = \frac{1}{2\sqrt{x}}$$

- If f is a function and c is a constant, then

  $$(cf)' = cf'$$

- If f and g are functions, then

  $$(f+g)' = f' + g'$$

  (Derivative is a linear operator)

Exercise 6-4: Evaluate the derivatives of the following functions:

a) $f(x) = 7x^3 + 3x^2 - 8$

b) $f(x) = 2\sqrt{x} - \frac{1}{4}x^4$

c) $f(x) = \frac{7x^3 - 18x + \sqrt{x}}{x}$

Exercise 6-5: Find the equation of the line that is tangent to $y = x^8 + x^4 + x^2$ at the point $(1, 3)$

Exercise 6-6: Find a horizontal tangent to

$$y = x^3 + x^2 - x + 1$$

Exercise 6-7: Find the equation of the tangent line to the graph at the given point. Then, sketch the function and the tangent line on the same coordinate system:

a) $f(x) = \frac{1}{x}$ at $(1, 1)$

b) $f(x) = \sqrt{x}$ at $(9, 3)$

c) $f(x) = x^3$ at $(0, 0)$

d) $f(x) = x^2$ at $(1, 1)$

e) $f(x) = x^3$ at $(1, 1)$

Differentiability and Continuity: If f is differentiable at a, then it is continuous at a, but if it is continuous at a, it is not necessarily differentiable.

If the graph of a function has no unique tangent at a point, then the function has no derivative at that point. For example, $f(x) = |x|$ is not differentiable at $x = 0$. (Its derivative does not exist)

Higher Order Derivatives: We can find the derivative of the derivative of a function. It is called second derivative and denoted by:

$$y'', \quad f''(x), \quad \frac{d^2y}{dx^2}$$

For third derivative, we use f''' but for fourth and higher derivatives, we use the notation $f^{(4)}(x)$

Exercise 6-8: Let $f(x) = 7x^3 - 18x$. Find f', f'' and f'''.

Vertical Tangents: The curve $y = f(x)$ has a vertical tangent at $(a, f(a))$ if f is continuous at a and if $|f'(x)| \to \infty$ as $x \to a$.

Exercise 6-9: Find the equation of the tangent line to $f(x) = \sqrt{x}$ at $(0, 0)$ and sketch the function and the tangent line on the same coordinate system.

Product Rule: If f and g are differentiable at x, then fg is differentiable at x and

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

or more briefly:

$$(fg)' = f'g + fg'$$

Reciprocal Rule: If f is differentiable at x and if $f(x) \neq 0$ then:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

Quotient Rule: If f and g are differentiable at x, and $g(x) \neq 0$ then $\dfrac{f}{g}$ is differentiable at x and:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Exercise 6-10: Evaluate the derivatives of the following functions:

a) $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

b) $f(x) = \dfrac{x^4 - 7x^2 + 5}{x}$

c) $f(x) = \dfrac{1}{1 - x^{-3}}$

d) $f(x) = \dfrac{x + \sqrt{x}}{x^2 - \sqrt{x}}$

Derivatives of Exponential and Trigonometric Functions:

We state without proof (at this stage) that:

$$\frac{de^x}{dx} = e^x$$

Actually, this is what make the number e so special. It is the only nonzero function whose derivative is itself.

The derivative of $\sin x$ is:

$$\frac{d\sin x}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$
$$= \sin x \lim_{h\to 0} \frac{1 - 2\sin^2(h/2) - 1}{h} + \cos x \cdot 1$$
$$= \cos x$$

We can show that

$$\frac{d \cos x}{dx} = -\sin x$$

similarly.

Exercise 6-11: Prove that

$$\frac{d \tan x}{dx} = \sec^2 x$$
$$\frac{d \cot x}{dx} = -\csc^2 x$$
$$\frac{d \sec x}{dx} = \sec x \tan x$$
$$\frac{d \csc x}{dx} = -\csc x \cot x$$

Exercise 6-12: Evaluate the derivatives of the following functions:

a) $y = \dfrac{x - \cos x}{x + \cos x}$

b) $y = x^n e^x$

c) $y = \sin 2\theta$

d) $s = \theta^2 \sec \theta$

e) $p = \dfrac{t^2 \sin t}{t - \tan t}$

Exercise 6-13: Find the following limits:

a) $\displaystyle\lim_{t\to\pi/4} \frac{\tan t - 1}{t - \pi/4}$

b) $\displaystyle\lim_{t\to\pi/3} \frac{\cos t - 1/2}{t - \pi/3}$

Chain Rule: If f and g are differentiable then $f(g(x))$ is also differentiable and

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

or more briefly

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Power Rule for Rational Powers: If r is a rational number and $f(x) = x^r$ then $f' = rx^{r-1}$
Let $r = \frac{p}{q}$. Then $y^q = x^p$.
Derivative of both sides gives

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{px^{p-1}}{qx^{\frac{p}{q}(q-1)}} = \frac{p}{q}\,x^{\frac{p}{q}-1}$$

Exercise 6-14: Find y'

a) $y = (x^2 - 2)^2$

b) $y = [x + (x^3 + 5)^2]^4$

c) $y = \left(t - \frac{1}{t}\right)^2$

d) $y = \sqrt{1 + \sqrt{1 + x}}$

e) $y = te^{-3t^2}$

f) $y = 3^t$

g) $y = 5^{3t}$

h) $y = e^{s^2(1-s)^3}$

i) $y = \cos^2\theta$

j) $y = 2\cos^3(5\theta)$

k) $y = \sin\big(\sin(\pi\theta)\big)$

Review Exercises

Exercise 6-15: Find the derivatives of the following functions:

a) $s = \frac{\sqrt{t}}{3 - 2\sqrt{t}}$

b) $y = x^5\cos^2(3x^4)$

c) $r = \tan\sqrt{3\theta - 2}$

d) $r = \left(\frac{\cos\theta}{1 - \theta\sin\theta}\right)^2$

e) $p = \frac{1}{(q - q^2 - \cos^2 q)^3}$

f) $y = x^{-5/3}\cos(3x - 1) - \sqrt{2}e^{x-\sqrt{2}}$

Exercise 6-16: Evaluate the derivatives of the following functions:

a) $f = \sin^2 \sqrt{x}$

b) $f = \dfrac{x}{\cos 4x}$

c) $f = \cos(\sin x)$

d) $f = \sqrt{\cos \sqrt{x}}$

e) $f = x^2 \cos \dfrac{1}{x}$

f) $f = \dfrac{1}{\sin^2 x + \cos^2 x}$

— End of WEEK —

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