Week 7 – Derivative - II

Implicit Differentiation: An equation involving x and y may define y as a function of x. This is called an implicit function.
For example,

$$x^2 + y^2 = 1, \quad ye^y + 2x - \cos y = 0$$

define y implicitly.

$$y = x^3 - 5x^2, \quad y = \cos(x^2 - e^x)$$

define y explicitly.

Exercise 7-1: We know that y is a function of x. Is it explicit or implicit function?

a) $y = x^3 + \sin x + xe^x$

b) $y = \dfrac{1}{1 + \sin^2(\pi x)}$

c) $3xy + x^2y^3 + x = 5$

d) $e^x + e^y = \tan(xy)$

The derivative of y can be found without solving for y. This is called implicit differentiation. The main idea is,

- Differentiate with respect to x
- Solve for y'

Exercise 7-2: Find the slope of the tangent line to the curves at the given points:

a) $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

b) $x^2 + 3xy - (x - y)^3 = 4$ at $(1, 1)$.

c) $x^4 + y^4 = 8xy$ at $(2, 2)$.

Exercise 7-3: Find $\dfrac{dy}{dx}$

a) $\sqrt{x} + \sqrt{y} = 1$

b) $xy = \tan xy$

c) $x^4 + y^4 = 2xy^3$

d) $\cos^3 x = \sin(x + y)$

e) $x^{2/3} + y^{2/3} = 5$

f) $(x^2 + y^2)^2 = 50xy$

Inverse Functions:
If f has an inverse, then

$$f\big(f^{-1}(x)\big) = x$$

Differentiating both sides and using chain rule, we obtain:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

Exercise 7-4: Using the information about derivative of $f = x^2$, find the derivative of $\sqrt{x}$.

Exercise 7-5: Using the information about derivative of $f = e^x$, find the derivative of $\ln x$.

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \quad x \neq 0$$

if $a > 0$ and $a \neq 1$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}, \quad x \neq 0$$
$$\frac{d}{dx} a^x = a^x \ln a$$

Logarithmic Differentiation:
Logarithm transforms products into sums. This helps in finding derivatives of some complicated functions. For example if

$$y = \frac{(x^3 + 1)(x^2 - 1)}{x^8 + 6x^4 + 1}$$

then

$$\ln y = \ln(x^3 + 1) + \ln(x^2 - 1) - \ln(x^8 + 6x^4 + 1)$$

$$\frac{y'}{y} = \frac{3x^2}{x^3 + 1} + \frac{2x}{x^2 - 1} - \frac{8x^7 + 24x^3}{x^8 + 6x^4 + 1}$$

This is called logarithmic differentiation.

Exercise 7-6: Find y'

a) $y = \ln \dfrac{x-1}{x+1}$

b) $y = \ln \ln x$

c) $y = e^{x - \ln x}$

d) $y = 2^{-x}$

e) $y = x \sin \ln x$

f) $y = x^x$

g) $y = x^{\ln x}$

h) $y = \ln x^{\ln x}$

i) $y = \left(\dfrac{\sqrt[3]{x^5 - x - 1}}{6x + 7} \right)^4$

Inverse Trigonometric Functions:

We have to restrict the ranges to find the inverses of trigonometric functions. Otherwise, they won't be one-to one.

Note that if

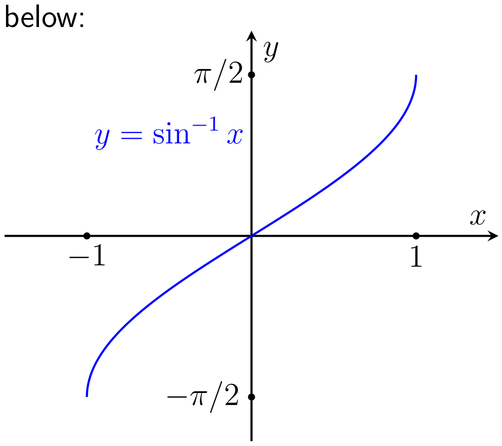
$$x = \tan y$$

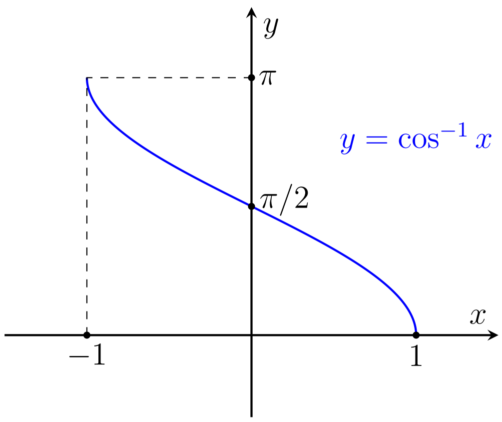
then

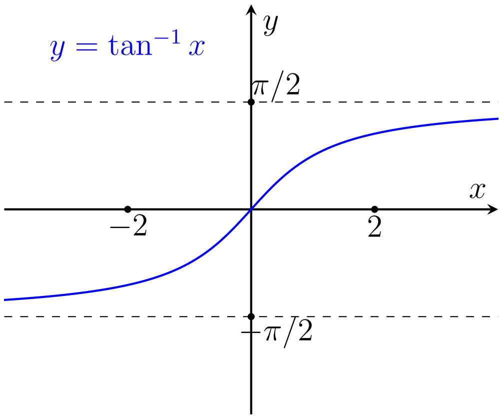
$$y = \arctan x \quad \text{or} \quad y = \tan^{-1} x$$

Function	Domain	Range
$\sin^{-1} x$	$-1 \leqslant x \leqslant 1$	$-\pi/2 \leqslant y \leqslant \pi/2$
$\cos^{-1} x$	$-1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$\tan^{-1} x$	$-\infty \leqslant x \leqslant +\infty$	$-\pi/2 < y < \pi/2$
$\cot^{-1} x$	$-\infty \leqslant x \leqslant +\infty$	$0 < y < \pi$
$\sec^{-1} x$	$\lvert x\rvert \geqslant 1$	$0 \leqslant y < \pi/2$, $\pi/2 < y \leqslant \pi$
$\csc^{-1} x$	$\lvert x\rvert \geqslant 1$	$-\pi/2 \leqslant y < 0$, $0 < y \leqslant \pi/2$

The graphs of some of these functions are given below:







Exercise 7-7: Find the following angles:

a) $\cos^{-1} \frac{1}{2}$

b) $\cos^{-1} \frac{1}{\sqrt{2}}$

c) $\csc^{-1} 2$

d) $\cot^{-1} \frac{-1}{\sqrt{3}}$

We can find the derivatives of the inverse trigonometric functions as follows:
If $y = \arctan x$ then $\tan y = x$. Evaluating the derivative of both sides, we find

$$(1 + \tan^2 y)\, y' = 1$$

$$\Rightarrow \quad y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Exercise 7-8: Find derivatives of $y = \arcsin x$, $y = \arccos x$ and $y = \sec^{-1} x$ using the same method.

Function	Derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert \sqrt{x^2-1}}$
$\csc^{-1} x$	$\frac{-1}{\lvert x\rvert \sqrt{x^2-1}}$

Exercise 7-9: Find y'

a) $y = \cos^{-1} x^3$

b) $y = \sin^{-1}(1-s)$

c) $y = \tan^{-1} \frac{1}{1+t}$

d) $y = \sec^{-1} \ln x$

e) $y = \csc^{-1} \frac{1}{x}$

f) $y = \cos^{-1} \sin x$

Review Exercises

Exercise 7-10: Find y'

a) $x^2y^2 + 8xy - 4x^4 = 20$

b) $xe^y + ye^x + xy = 72$

c) $y^3 + y = 2\cos x$

Exercise 7-11: Find y' at the indicated point:

a) $x^3 + y^3 = 9xy$ at $(2, 4)$

b) $5x^{4/5} + 10y^{6/5} = 15$ at $(1, 1)$

Exercise 7-12: Find the equation of the tangent line to $x + \sqrt{xy} = 6$ at $(4, 1)$

Exercise 7-13: Find y'

a) $y = \ln\left(\dfrac{\sin^2\theta}{\theta e^\theta}\right)$

b) $y = s^{\sqrt{s}-1}$

c) $y = 5^{x-3\ln x}$

d) $y = \left(\dfrac{x-2}{(3+x)(1+x^2)}\right)^x$

e) $y = \arcsin(e^t)$

f) $y = \arccos\dfrac{1}{1-\ln t}$

g) $y = e^{\arctan x}$

h) $y = \sin\sec^{-1} x$

i) $y = \cos\tan^{-1} x$

j) $y = \ln^3(x^2+1)$

— End of WEEK —

Author: Dr. Emre Sermutlu

Last Update: November 9, 2016