Week 8 – Extreme Values

Increments and Differentials

The change in the value of y is the increment Δy.

$$\Delta y = f(x + \Delta x) - f(x)$$

Differential is defined as

$$dy = f'(x)\,\Delta x$$

dy is an approximation to Δy. If Δx is small, approximation is good.

Suppose we know function and derivative values at $x = a$, we want to estimate f at a nearby point $x = a + \Delta x$.

$$f(a + \Delta x) = f(a) + \Delta y$$

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x$$

Now, replace Δx by $x - a$ to obtain

$$f(x) \approx f(a) + f'(a)(x - a)$$

This is the linear approximation to f near a. The approximation becomes better as x gets closer to a.

Exercise 8-1: Find a linear approximation of

$$f(x) = (1 + x)^n$$

near $x = 0$

Solution:

$$f'(x) = n(1 + x)^{n-1}$$

$$f(0) = 1, \qquad f'(0) = n$$

Linear approximation is:

$$1 + nx$$

In other words

$$(1 + x)^n \approx 1 + nx$$

The approximation becomes better as $x \to 0$.

For example,

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x$$

Exercise 8-2: Find a linear approximation of $f(x) = \sin x$ near

a) $x = 0$

b) $x = \pi$

Exercise 8-3: Estimate the following numbers without a calculator:

a) $\sqrt{98}$

b) $\sqrt[4]{15}$

c) $\cos 43^0$

d) 1.02^{12}

e) $\sqrt{1.03}$

Exercise 8-4: Write dy in terms of x and dx

a) $y = x + \sqrt{x^2+1}$

b) $y = \dfrac{x+1}{x^2-2}$

c) $y = \dfrac{\cos x}{\sqrt{x}}$

Absolute Maximum and Minimum Values

If

$$f(c) \leqslant f(x)$$

for all x on a set S of real numbers, $f(c)$ is the absolute minimum value of f on S. Similarly if

$$f(c) \geqslant f(x)$$

for all x on S, $f(c)$ is the absolute maximum value of f on S.

Theorem: If the function f is continuous on the closed interval $[a, b]$, then f has a maximum and a minimum value on $[a, b]$.

(Note that, if f is not continuous or if the interval is not closed, there may or may not be extreme values.)

Local Extrema: $f(c)$ is local maximum if

$$f(x) \leqslant f(c)$$

for all x in some open interval containing c.
$f(c)$ is local minimum if

$$f(x) \geqslant f(c)$$

for all x in some open interval containing c.

Critical Point: A number c is called a critical point of f if $f'(c) = 0$ or $f'(c)$ does not exist.
Note that f can have a local extremum only on a critical point.

Theorem: Suppose that f is continuous and $f(c)$ is the absolute maximum (or minimum) of f on $[a, b]$. Then c is either a critical point of f or an endpoint.

How to find absolute extrema:

- Find the points where $f' = 0$
- Find the points where f' does not exist.
- Consider such points only if they are **inside** the given interval.
- Consider endpoints.
- Check all candidates. Both absolute minimum and maximum are among them.

Exercise 8-5: Find the maximum and minimum values of f on the given interval:

a) $f(x) = 12 - x^2$ on $[2, 4]$

b) $f(x) = 12 - x^2$ on $[-2, 4]$

c) $f(x) = x^3 - 2x$ on $[-1, 4]$

d) $f(x) = x + \dfrac{9}{x}$ on $[1, 4]$

e) $f(x) = 3x^5 - 5x^3$ on $[-2, 2]$

f) $f(x) = |3x - 5|$ on $[0, 2]$

g) $f(x) = x\sqrt{1 - x^2}$ on $[-1, 1]$

Exercise 8-6: Find two non-negative real numbers such that their sum is 40 and their product is as large as possible.

Rolle's Theorem: Let f be continuous and differentiable on $[a, b]$. If $f(a) = f(b)$, then there exists c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem: Let f be continuous and differentiable on $[a, b]$. There exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollaries of the Mean Value Theorem:

- If $f' = 0$ on $[a, b]$ then f is constant on $[a, b]$. $(f' = 0 \quad \Rightarrow \quad f = c)$
- If the derivatives of two functions are equal on $[a, b]$, then they differ by a constant on $[a, b]$. $(f' = g' \quad \Rightarrow \quad f - g = c)$
- If $f' > 0$ then f is increasing. If $f' < 0$ then f is decreasing.

Increasing and Decreasing Functions: If

$$f(x_1) < f(x_2) \text{ for all } x_1 < x_2$$

f is increasing. If

$$f(x_1) > f(x_2) \text{ for all } x_1 > x_2$$

f is decreasing.

First Derivative Test for Local Extrema: Let f be continuous on I and differentiable there except possibly on c. f has a local extremum at c if and only if f' changes sign at c.

- If $f' < 0$ for $x < c$ and $f' > 0$ for $x > c$ then $f(c)$ is a local minimum.
- If $f' > 0$ for $x < c$ and $f' < 0$ for $x > c$ then $f(c)$ is a local maximum.

Exercise 8-7: Determine the intervals on which f is increasing and decreasing:

a) $f(x) = 16 - 4x^2$

b) $f(x) = x^4 - 2x^2 + 1$

c) $f(x) = \dfrac{x}{x+1}$

Review Exercises

Exercise 8-8: Estimate the following without a calculator. Then, compare your estimation with exact results.

a) $\dfrac{1}{\sqrt{1.008}}$

b) $(99.7)^{3/2}$

c) $\sqrt{26}$

Exercise 8-9: Find the linearization of $f = \tan x$ at $x = \pi/3$ and $x = \pi/4$. Sketch the function and linearization.

Exercise 8-10: Find and classify the critical points of the following functions:

a) $f(x) = (x^2 - 3)e^x$

b) $f(x) = \dfrac{x^3}{3} - 2x^2 + 4x$

c) $f(x) = x^{1/3}(x - 4)$

Exercise 8-11: Find two non-negative real numbers x, y such that

$$2x + 3y = 60$$

and their product xy is as large as possible.

Exercise 8-12: Find the maximum and minimum values of f on the given interval:

a) $f = x^{2/3}$ on $[-2, 3]$

b) $f = 10x(2 - \ln x)$ on $[1, e^2]$

Exercise 8-13: Determine the intervals where the following functions are increasing and decreasing:

a) $f(x) = x^3 - 12x - 5$

b) $f(x) = \dfrac{x^2 - 3}{x - 2}$

c) $f(x) = 4x^5 + 5x^4 - 40x^3$

d) $f(x) = x^4 e^{-x}$

— End of WEEK —

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