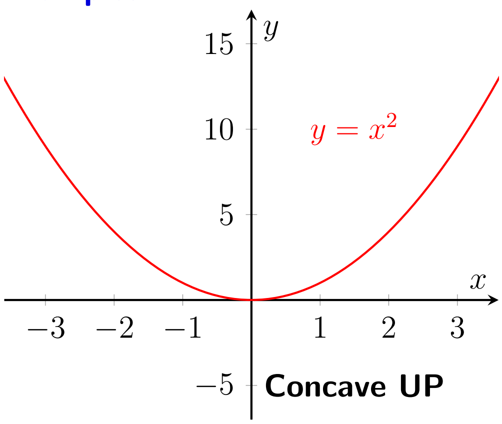
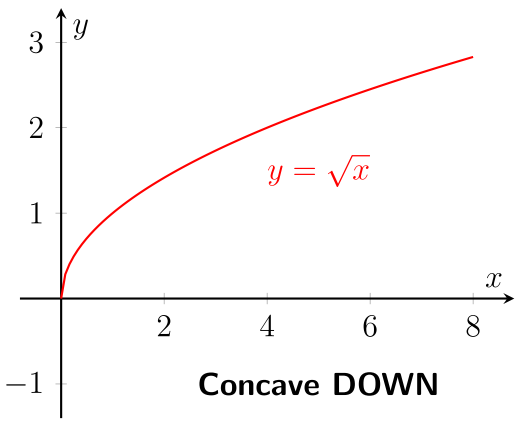
Week 9 – Curve Sketching

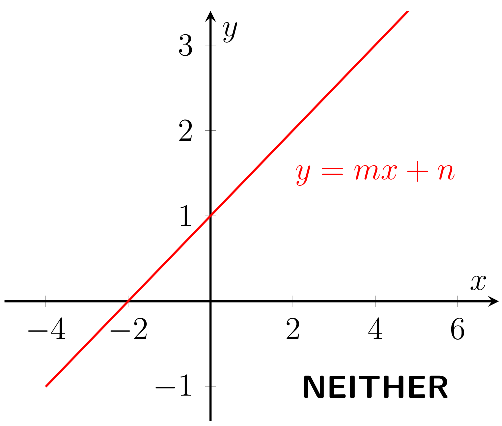
Concavity: The graph of a differentiable function is concave up if f' increasing, it is concave down if f' decreasing.

Test for Concavity: If $f''(x) > 0$, then f is concave up at x. If $f''(x) < 0$, then f is concave down at x.

Examples:







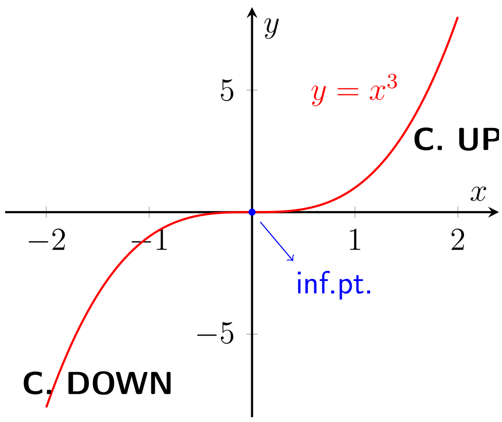
Inflection Point: An inflection point is a point where the concavity changes. In other words, If a is an inflection point, $f'' > 0$ on one side of a and $f'' < 0$ on the other side. This means either $f''(a) = 0$ or $f''(a)$ does not exist.

Exercise 9-1: Determine the concavity of

$$f(x) = x^3$$

Solution: $f' = 3x^2$, $f'' = 6x$. For $x > 0$, it is concave up, for $x < 0$ it is concave down. $x = 0$ is the inflection point.

x		0	
f''	$-$	0	$+$
f is:	concave down		concave up



Exercise 9-2: For the following functions, determine the concavity, find the inflection points and sketch their graphs:

a) $f(x) = x^4$

b) $f(x) = \sqrt{|x|}$

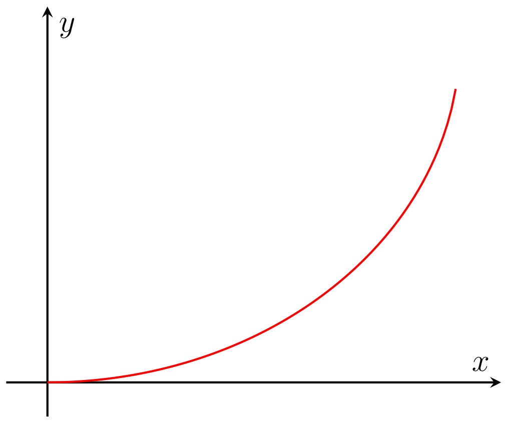
c) $f(x) = \ln x$

d) $f(x) = xe^{-x}$

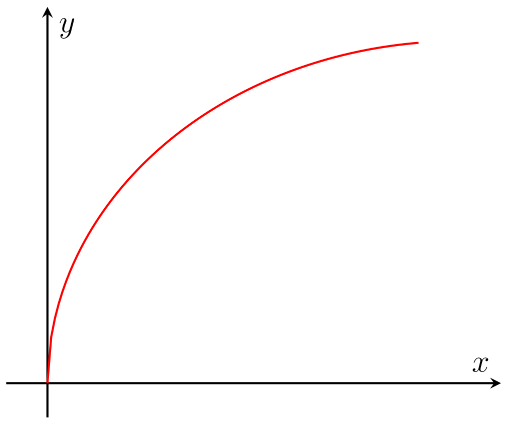
e) $f(x) = \sin x, \quad x \in [-\pi, \pi]$

Shape of a graph based on f' and f'':

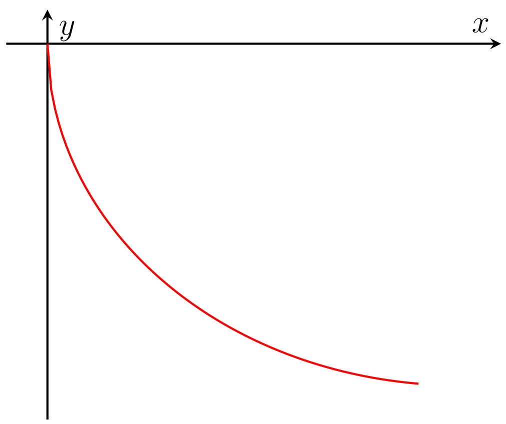
- : $f' > 0, f'' > 0$



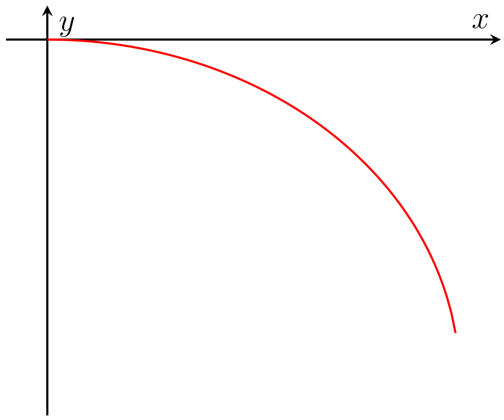
- : $f' > 0, f'' < 0$



- : $f' < 0, f'' < 0$



- : $f' < 0, f'' < 0$



Second Derivative Test: Suppose that f is twice differentiable at c and $f'(c) = 0$. Then

- If $f''(c) > 0$, f has a local minimum at c.
- If $f''(c) < 0$, f has a local maximum at c.
- If $f''(c) = 0$ then the test is inconclusive. f may have a local minimum, local maximum or neither at $x = c$.

How to Sketch a Graph:

- Identify domain of f, symmetries, x and y intercepts. (if any)
- Find first and second derivatives of f.
- Find critical points, inflections points.
- Find asymptotes.
- Make a table and include all this information.
- Sketch the curve using the table.

Exercise 9-3: Graph $f = x^3 + 3x^2 - 24x$

Solution:

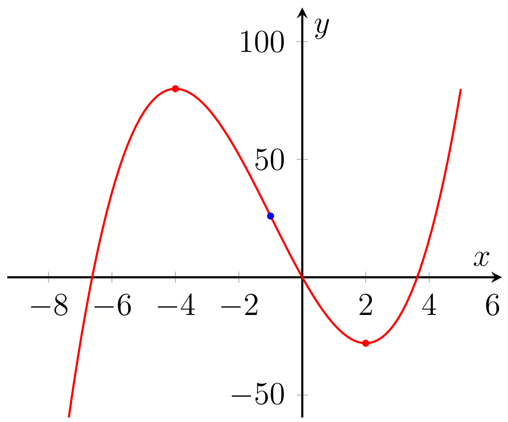
$$\lim_{x \to \infty} f = +\infty, \lim_{x \to -\infty} f = -\infty$$

$$f' = 3x^2 + 6x - 24$$
$$= 3(x+4)(x-2)$$
$$f' = 0 \quad \Rightarrow \quad x = -4, \quad \text{and} \quad x = 2$$
$$f'' = 6x + 6$$
$$f'' = 0 \quad \Rightarrow \quad x = -1$$

Also, some specific points are:

$$f(-4) = 80, \quad f(-1) = 26$$
$$f(0) = 0, \quad f(2) = -28$$

x		-4		-1		2	
f'	$+$	0	$-$		$-$	0	$+$
f''	$-$		$-$	0	$+$		$+$
f	↗		↘		↘		↗



Exercise 9-4: Sketch the graph of the following functions:

a) $f(x) = 12x^5 + 30x^4 - 25x^3$

b) $f(x) = 3x^4 + 4x^3 - 36x^2 + 40$

c) $f(x) = x^{1/3}(4 - x)$

d) $f(x) = x\sqrt{3 - x}$

Exercise 9-5: Sketch the graph of

$$f(x) = \frac{3x - 8}{x + 4}$$

Solution:

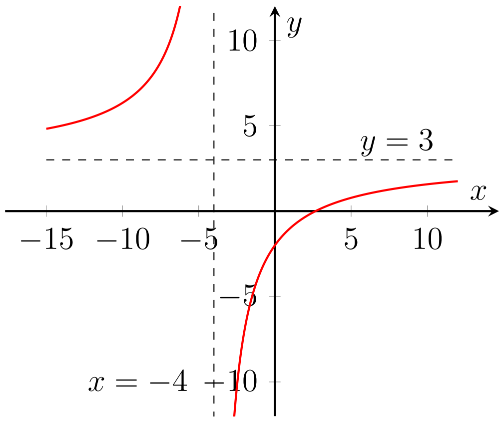
$$f' = \frac{20}{(x+4)^2}$$
$$f'' = \frac{-40}{(x+4)^3}$$

$\lim_{x\to\infty} f = 3 \quad \Rightarrow \quad y = 3$ is Horizontal Asymptote.

$\lim_{x\to 4^+} f = \infty$ and $\lim_{x\to 4^-} f = -\infty \quad \Rightarrow$
$x = -4$ is Vertical Asymptote.

f' or f'' are never zero. They are undefined at $x = -4$.

x		-4	
f'	$+$	‖	$+$
f''	$+$	‖	$-$
f	↗	‖	↗



Exercise 9-6: Sketch the graphs of the following functions.

a) $f(x) = \dfrac{1}{(x-8)^2}$

b) $f(x) = \dfrac{x^2}{x-1}$

Review Exercises

Exercise 9-7: Sketch the graphs of the following functions.

a) $f(x) = 3x^4 - 4x^3 - 5$

b) $f(x) = (x-1)^2(x+2)^3$

c) $f(x) = x^{1/3}(6-x)^{2/3}$

d) $f(x) = \dfrac{1}{x^2 - x - 2}$

e) $f(x) = \dfrac{2x^3 - 5x^2 + 4x}{x^2 - 2x + 1}$

f) $f(x) = \dfrac{4 - x^3}{x^2}$

— End of WEEK —

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