Çankaya University
Department of Mathematics
2016 - 2017 Fall Semester

MATH 155 - Calculus for Engineering I

Final Examination

1) Evaluate the following limits if they exist.

a)(7 pts)

$$\lim_{x\to 0} \frac{\tan(3x)}{\sqrt{x}\sin(2\sqrt{x})}$$

b)(8 pts)

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3 + x^4}$$

2) a)(9 pts) Find the derivative of the function:

$$y = f(x) = (x + e^x)^{\ln x}$$

b)(6 pts) Find the slope of the tangent line to the curve $x^8 + 4x^2y^2 + y^8 = 6$ at the point $(1, 1)$.

3) Evaluate the integrals:

a)(7 pts)

$$\int (x^3 + 6x^2)^7 (x^2 + 4x) dx$$

b)(8 pts)

$$\int_1^2 \frac{3e^{3x} + 2}{e^{3x} + 2x} dx$$

4) a)(10 pts) Find the area between the curve $y = x^2 - 5x + 4$ and the line $y = 18$.

b)(10 pts) Evaluate the integral

$$\int (3x+1)e^{-2x}dx$$

5) Evaluate the integrals.

a)(10 pts)

$$\int_{\frac{\pi}{2}}^{2\pi} \sin^3 x \cos^{12} x \, dx$$

b)(10 pts)

$$\int \frac{dx}{\sqrt{7+x^2}}$$

6) a)(9 pts) Evaluate the integral

$$\int \frac{4x^2+2x-2}{(x^2+1)(2x-3)}\, dx$$

b)(6 pts) Evaluate the integral if exists

$$\int_0^\infty e^{-2x}dx$$

Answers

1) a) Using the change of variable $u = \sqrt{x}$ we obtain:

$$\lim_{u \to 0} \frac{\tan(3u^2)}{u \sin(2u)}$$

Now multiply and divide by $3u$ and rearrange to obtain:

$$= \lim_{u \to 0} \frac{\tan(3u^2)}{3u^2} \cdot \frac{3u}{\sin(2u)}$$

Similarly, multiply and divide by 2 and rearrange:

$$= \lim_{u \to 0} \frac{3}{2} \cdot \frac{\tan(3u^2)}{3u^2} \cdot \frac{2u}{\sin(2u)}$$

$$= \frac{3}{2} \cdot 1 \cdot 1 = \frac{3}{2}$$

b)This limit is in the form $\frac{0}{0}$, so using L'Hôpital's rule we obtain:

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3 + x^4} &= \lim_{x \to 0} \frac{e^x - 1 - x}{3x^2 + 4x^3} \\ &= \lim_{x \to 0} \frac{e^x - 1}{6x + 12x^2} \\ &= \lim_{x \to 0} \frac{e^x}{6 + 24x} \end{aligned}$$

At this point, the limit is NOT in the form $\frac{0}{0}$, so we can NOT use L'Hôpital.

Just insert $x = 0$ to obtain:

$$= \frac{1}{6}$$

2) a)

$$\begin{aligned}
\ln y &= \ln x \ln(x+e^x) \\
(\ln y)' &= \frac{1}{x}\ln(x+e^x) + \frac{1+e^x}{x+e^x}\ln x \\
\frac{y'}{y} &= \frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x}\ln x \\
y' &= (x+e^x)^{\ln x}\left(\frac{\ln(x+e^x)}{x} + \frac{1+e^x}{x+e^x}\ln x\right)
\end{aligned}$$

b) Using implicit differentiation we obtain:

$$8x^7 + 8xy^2 + 8x^2yy' + 8y^7y' = 0$$

$$y' = \frac{-x^7 - xy^2}{x^2y + y^7}$$

At $(1,1)$ the slope is:

$$y' = \frac{-2}{2} = -1$$

3) a) The substitution $u = x^3 + 6x^2$ gives

$$\begin{aligned} du &= (3x^2 + 12x)\,dx \\ \frac{1}{3}\,du &= (x^2 + 4x)\,dx \end{aligned}$$

Rewriting the integral in terms of u, we obtain:

$$\begin{aligned} \int (x^3 + 6x^2)^7 (x^2 + 4x)\,dx &= \frac{1}{3}\int u^7\,du \\ &= \frac{u^8}{24} + c \\ &= \frac{(x^3 + 6x^2)^8}{24} + c \end{aligned}$$

b) Use the substitution

$$u = e^{3x} + 2x \quad \Rightarrow \quad du = (3e^{3x} + 2)\,dx$$

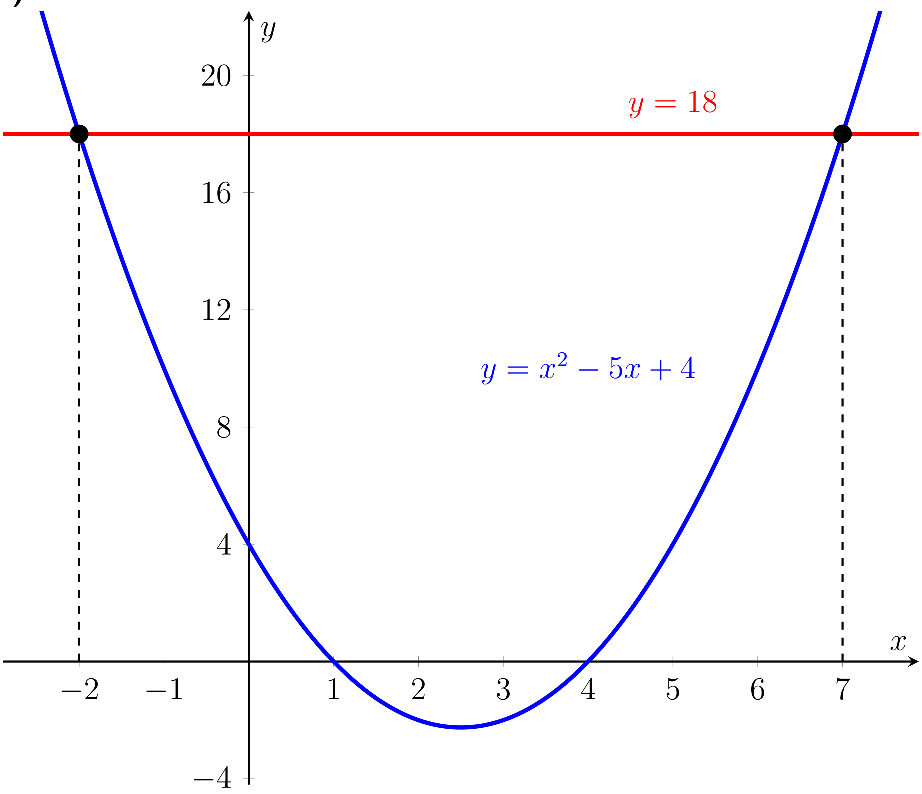
The new integral limits are:

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad u = e^3 + 2 \\ x = 2 \quad &\Rightarrow \quad u = e^6 + 4 \end{aligned}$$

Rewriting the integral in terms of u, we obtain:

$$\begin{aligned} \int_1^2 \frac{3e^{3x} + 2}{e^{3x} + 2x}\,dx &= \int_{e^3+2}^{e^6+4} \frac{du}{u} \\ &= \ln|u|\,\Big|_{e^3+2}^{e^6+4} \\ &= \ln(e^6 + 4) - \ln(e^3 + 2) \\ &= \ln\frac{e^6 + 4}{e^3 + 2} \end{aligned}$$

4) a)



$$x^2 - 5x + 4 = 18$$

$$x^2 - 5x - 14 = 0$$

$$(x-7)(x+2) = 0 \quad \Rightarrow \quad x = 7, \quad x = -2$$

The area is:

$$\begin{aligned}
A &= \int_{-2}^{7} \Big(18 - (x^2 - 5x + 4)\Big)\, dx \\
&= \int_{-2}^{7} \Big(14 + 5x - x^2\Big)\, dx \\
&= 14x + \frac{5}{2}x^2 - \frac{x^3}{3}\Bigg|_{-2}^{7} \\
&= \left(98 + \frac{245}{2} - \frac{343}{3}\right) - \left(-28 + 10 + \frac{8}{3}\right) \\
&= \frac{243}{2}
\end{aligned}$$

b) We have to use integration by parts.

$$\begin{aligned} u &= 3x+1 &\Rightarrow\quad du &= 3\,dx \\ dv &= e^{-2x}\,dx &\Rightarrow\quad v &= \frac{e^{-2x}}{-2} \end{aligned}$$

$$\begin{aligned} \int (3x+1)e^{-2x}dx &= (3x+1)\,\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\,3\,dx \\ &= -\frac{(3x+1)\,e^{-2x}}{2} + \frac{3}{2}\int e^{-2x}\,dx \\ &= -\frac{(3x+1)\,e^{-2x}}{2} - \frac{3}{4}e^{-2x} + c \\ &= -\frac{(6x+5)\,e^{-2x}}{4} + c \end{aligned}$$

5) a) $$\begin{aligned} \int_{\frac{\pi}{2}}^{2\pi} \sin^3 x\cos^{12} x\,dx &= \int_{\frac{\pi}{2}}^{2\pi} \sin^2 x\cos^{12} x\sin x\,dx \\ &= \int_{\frac{\pi}{2}}^{2\pi} (1-\cos^2 x)\cos^{12} x\sin x\,dx \end{aligned}$$

Using the substitution: $u = \cos x,\ du = -\sin x\,dx$

$$\begin{aligned} &= \int_0^1 -(1-u^2)u^{12}\,du \\ &= \int_0^1 (u^{14}-u^{12})\,du \\ &= \frac{u^{15}}{15} - \frac{u^{13}}{13}\Bigg|_0^1 \\ &= -\frac{2}{195} \end{aligned}$$

b) We need the trigonometric substitution

$$x = \sqrt{7}\,\tan\theta, \quad dx = \sqrt{7}\,\sec^2\theta\,d\theta$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{7+x^2}} &= \int \frac{\sqrt{7}\sec^2\theta\,d\theta}{\sqrt{7+7\tan^2\theta}} \\
&= \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} \\
&= \int \frac{\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}} \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + c \\
&= \ln\left|\sec\left(\arctan\frac{x}{\sqrt{7}}\right) + \frac{x}{\sqrt{7}}\right| + c \qquad \checkmark
\end{aligned}$$

OR:

$$\begin{aligned}
&= \ln\left|\sqrt{1+\frac{x^2}{7}} + \frac{x}{\sqrt{7}}\right| + c \qquad \checkmark \\
&= \ln\left|\sqrt{7+x^2}+x\right| - \ln\sqrt{7} + c
\end{aligned}$$

OR:

$$= \ln\left|\sqrt{7+x^2}+x\right| + k \qquad \checkmark$$

6) a) Using partial fractions expansion, we find:

$$\frac{4x^2+2x-2}{(x^2+1)(2x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{(2x-3)}$$

$$4x^2+2x-2 = (Ax+B)(2x-3) + C(x^2+1)$$

$$4x^2+2x-2 = (2A+C)x^2 + (-3A+2B)x + (-3B+C)$$

$$\Rightarrow \quad \begin{aligned} 2A+C &= 4 \\ -3A+2B &= 2 \\ -3B+C &= -2 \end{aligned} \quad \Rightarrow \quad A = \frac{6}{13},\ B = \frac{22}{13},\ C = \frac{40}{13}$$

$$\int \frac{4x^2+2x-2}{(x^2+1)(2x-3)}\,dx = \frac{6}{13}\int \frac{x\,dx}{x^2+1} + \frac{22}{13}\int \frac{dx}{x^2+1} + \frac{40}{13}\int \frac{dx}{2x-3}$$

$$= \frac{3}{13}\ln|x^2+1| + \frac{22}{13}\arctan x + \frac{20}{13}\ln\left|x-\frac{3}{2}\right| + c$$

b)
$$\begin{aligned} \int_0^\infty e^{-2x}dx &= \lim_{t\to\infty} \int_0^t e^{-2x}dx \\ &= \lim_{t\to\infty} \left.\frac{e^{-2x}}{-2}\right|_0^t \\ &= \lim_{t\to\infty} -\frac{e^{-2t}}{2} + \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

The integral is convergent.