Çankaya University
Department of Mathematics
2017 - 2018 Fall Semester

MATH 155 - Calculus for Engineering I

Final Examination

1) Evaluate the following limits if they exist.

a) $\displaystyle\lim_{x\to 0}\frac{x\sin^2(3x)}{\tan^6(2\sqrt{x})}$

b) $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1-2x}{3x^2}$

c) $\displaystyle\lim_{x\to\infty} x\sin x$

2) Evaluate the following integrals:

a) $\displaystyle\int_0^{\pi/2}\sin^3\theta\cos^{11}\theta\,d\theta$

b) $\displaystyle\int\frac{\ln x\,dx}{x\sqrt{1+(\ln x)^2}}$

3) a) Evaluate the integral $\displaystyle\int_0^1 2x(x^2+1)^2 e^{x^2+1}\,dx$

b) Find the area of the finite region bounded by the curve $y=e^x$, the line $x=0$ and the line tangent to $y=e^x$ at $(1,e)$.

4) Evaluate the following integrals:

a) $\displaystyle\int\frac{dy}{\sqrt{4y^2-4y-3}}$

b) $\displaystyle\int\frac{\sqrt{4u^2-1}}{u^2}du$

5) a) Evaluate $\displaystyle\int\frac{8x\,dx}{x^3-2x^2+4x-8}$

b) Is the improper integral $\displaystyle\int_1^{\infty}\frac{3+4e^{-x}}{x}dx$ convergent or divergent? Explain.

6) a) Evaluate the derivative $\displaystyle\frac{d}{dx}\int_x^{155}\frac{\sin t}{t}dt$ where $x>0$.

b) Evaluate the derivative of $f(x)=x^{\cos(x^2)}$.

c) Give an example of a function that is continuous at $x=0$ but is not differentiable at $x=0$.

Answers

1) a)
$$\lim_{x\to 0}\frac{x\sin^2(3x)}{\tan^6(2\sqrt{x})} = x\cdot\left(\frac{\sin(3x)}{3x}\right)^2\cdot 9x^2\cdot\left(\frac{2\sqrt{x}}{\tan(2\sqrt{x})}\right)^6\cdot\frac{1}{2^6x^3}$$
$$= \lim_{x\to 0} x\cdot 1\cdot 9x^2\cdot 1\cdot\frac{1}{64x^3}$$
$$= \frac{9}{64}$$

b) This limit is in the form $\frac{0}{0}$, so using L'Hôpital's rule we obtain:

$$\lim_{x\to 0}\frac{e^{2x}-1-2x}{3x^2} = \lim_{x\to 0}\frac{2e^{2x}-2}{6x}$$
$$= \lim_{x\to 0}\frac{4e^{2x}}{6}$$
$$= \frac{2}{3}$$

c) $\sin x$ takes on all the values on $[-1,1]$. If we multiply it by x, it will take on arbitrarily large and small values. Limit does not exist.

2) a) Using the substitution $u = \cos\theta \quad \Rightarrow \quad du = -\sin\theta\, d\theta$, we obtain the integration limits $u = 1$ and $u = 0$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^3\theta \cos^{11}\theta \, d\theta &= \int_0^{\pi/2} \sin^2\theta \cos^{11}\theta \, \sin\theta \, d\theta \\
&= \int_0^{\pi/2} \left(1 - \cos^2\theta\right) \cos^{11}\theta \, \sin\theta \, d\theta \\
&= \int_1^0 (1 - u^2) u^{11} \left(-\, du\right) \\
&= \int_0^1 \left(u^{11} - u^{13}\right) du \\
&= \frac{u^{12}}{12} - \frac{u^{14}}{14} \Bigg|_0^1 \\
&= \frac{1}{12} - \frac{1}{14} \\
&= \frac{1}{84}
\end{aligned}$$

b) Using the substitution $u = \ln x \quad \Rightarrow \quad du = \dfrac{dx}{x}$, we obtain:

$$\begin{aligned}
\int \frac{\ln x \, dx}{x\sqrt{1 + \left(\ln x\right)^2}} &= \int \frac{u\, du}{\sqrt{1 + u^2}} \\
&= \frac{1}{2} \frac{\left(1 + u^2\right)^{1/2}}{1/2} + c \\
&= \sqrt{1 + u^2} + c \\
&= \sqrt{1 + \left(\ln x\right)^2} + c
\end{aligned}$$

3) a) Let $I = \int_0^1 2x\left(x^2+1\right)^2 e^{x^2+1}\,dx$

Using the substitution $\quad y = x^2+1 \quad \Rightarrow \quad dy = 2x\,dx, \quad$ we obtain:

$$I = \int_1^2 y^2 e^y\,dy$$

Now use integration by parts

$u = y^2, \quad dv = e^y\,dy$

$\Rightarrow \quad du = 2y\,dy, \quad v = e^y$

$$I = y^2e^y\Big|_1^2 - \int_1^2 2ye^y\,dy$$

One more integration by parts gives:

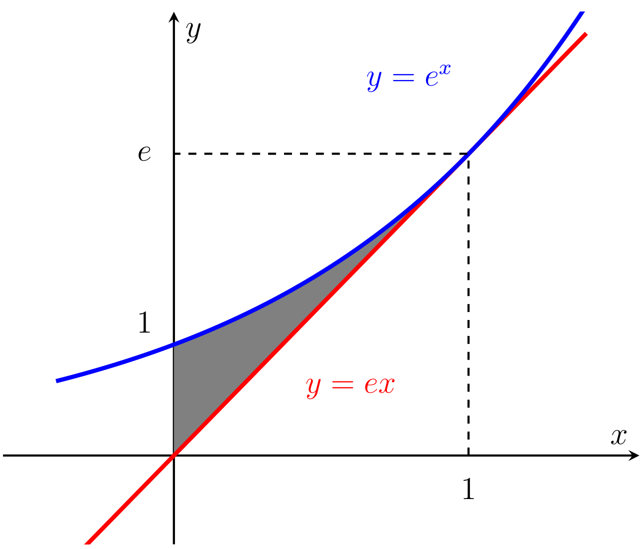
$$\begin{aligned} I &= y^2e^y\Big|_1^2 - 2ye^y\Big|_1^2 + \int_1^2 2e^y\,dy \\ &= y^2e^y - 2ye^y + 2e^y\Big|_1^2 \\ &= \left(4e^2-4e^2+2e^2\right) - \left(e^1-2e^1+2e^1\right) \\ &= 2e^2 - e \end{aligned}$$

b) $y = e^x \quad \Rightarrow \quad y' = e^x, \quad y'(1) = e$

The tangent line is: $y - e = e(x-1) \quad \Rightarrow \quad y = ex$

The shaded area is:

$$\begin{aligned} A &= \int_0^1 \left(e^x - ex\right)dx \\ &= e^x - e\frac{x^2}{2}\Big|_0^1 \\ &= e - \frac{e}{2} - 1 + 0 \\ &= \frac{e}{2} - 1 \end{aligned}$$



4) a) Using the substitution $u = 2y - 1 \quad \Rightarrow \quad du = 2dy,$ we obtain:

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{\frac{1}{2}\,du}{\sqrt{u^2 - 4}}$$

$$u = 2\sec\theta \quad \Rightarrow \quad du = 2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta\,d\theta}{2\tan\theta}$$

$$= \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{u}{2} + \sqrt{\frac{u^2}{4} - 1}\,\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2y - 1 + \sqrt{4y^2 - 4y - 3}}{2}\,\right| + c$$

$$= \frac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\,\right| + c_2$$

b) Using the substitution $2u = \sec\theta \quad \Rightarrow \quad du = \frac{1}{2}\sec\theta\tan\theta\,d\theta,$ we obtain:

$$\int \frac{\sqrt{4u^2 - 1}}{u^2}du = \int \frac{4\tan\theta}{\sec^2\theta}\,\frac{1}{2}\sec\theta\tan\theta\,d\theta$$

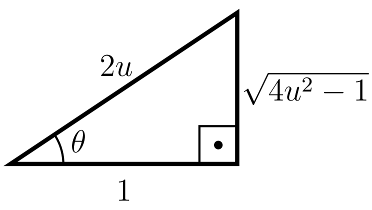
$$= 2\int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= 2\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= 2\left(\int \sec\theta\,d\theta - \int \cos\theta\,d\theta\right)$$

$$= 2\Big(\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\Big) + c$$

$$= 2\ln\left(2u + \sqrt{4u^2 - 1}\right) - \frac{\sqrt{4u^2 - 1}}{u} + c$$



$$\sec\theta = 2u$$

$$\tan\theta = \sqrt{4u^2 - 1}$$

$$\sin\theta = \frac{\sqrt{4u^2 - 1}}{2u}$$

5)
$$\begin{aligned}\int \frac{8x\,dx}{x^3-2x^2+4x-8} &= \int \frac{8x\,dx}{(x^2+4)(x-2)} \\ &= \int \left(\frac{Ax+B}{x^2+4}+\frac{C}{x-2}\right)dx \\ &= \int \left(\frac{-2x}{x^2+4}+\frac{4}{x^2+4}+\frac{2}{x-2}\right)dx \\ &= -\ln\left(x^2+4\right)+2\arctan\left(\frac{x}{2}\right)+2\ln|x-2|+c\end{aligned}$$

b) We know that $\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent.

$$\begin{aligned} e^{-x} &> 0 \\ 4e^{-x} &> 0 \\ 3+4e^{-x} &> 3 \end{aligned}$$

Therefore $\displaystyle\int_1^\infty \frac{3+4e^{-x}}{x}\,dx$ is divergent by direct comparison test.

6) a) Using the formula $\displaystyle\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u)\,u' - f(v)\,v'$

we obtain: $\displaystyle\frac{d}{dx}\int_x^{155}\frac{\sin t}{t}\,dt = -\frac{\sin x}{x}$

b) We will use logarithmic differentiation:

$$\begin{aligned} f(x) &= x^{\cos(x^2)} \\ \ln f(x) &= \cos(x^2)\ln x \\ \frac{f'(x)}{f(x)} &= -2x\sin(x^2)\ln x+\frac{\cos(x^2)}{x} \\ f'(x) &= x^{\cos(x^2)}\left(-2x\sin\left(x^2\right)\ln x+\frac{\cos\left(x^2\right)}{x}\right)\end{aligned}$$

c) The absolute value function $y=\left|x\right|$ is continuous at $x=0$ but is not differentiable at $x=0$.