Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 155 - Calculus for Engineering I

Final Examination

1) a) Does the function $f(x) = |x-4| - 2$ have absolute maximum and minimum values on the interval $\big[0,\, 10\big]$? If it has, find them.

b) Evaluate the limit $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\ln\big(\csc(x)\big)}{\big(x-\frac{\pi}{2}\big)^2}$.

2) a) Find $\dfrac{dy}{dx}$ at $x=1,\ y=\dfrac{1}{2}$ using the equation $\arcsin\big(xy\big) + y\tan\big(\ln(x)\big) = \dfrac{\pi}{6}$.

(Here, y is a function of x, in other words $y = y(x)$)

b) Evaluate the integral $\displaystyle\int \tan^5 x \, \sec^6 x \, dx$.

3) a) Evaluate the integral $\displaystyle\int \frac{e^x\,dx}{9+e^{2x}}$.

b) Find the area of the region bounded by the curves $y = e^x,\ y = e^{-2x}$ and the line $x = 2$.

4) a) Evaluate the integral $\displaystyle\int \arcsin\big(5x\big)dx$.

b) Evaluate the integral $\displaystyle\int_0^{\frac{1}{3}} \sin\big(\pi x\big)\cos\big(3\pi x\big)dx$.

c) Evaluate the integral $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$ if it is convergent.

5) Evaluate the integral $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-2x}}$.

Bonus)

Rolle's Theorem: Suppose $f(x)$ is a function that is continuous on $\big[a,\, b\big]$ and differentiable on $\big(a,\, b\big)$. If $f(a) = f(b)$ then there exists a number $c \in \big(a,\, b\big)$ such that $f'(c) = 0$.

Decide whether there exists a point $c \in \mathbb{R}$ such that $f'(c) = 0$ where

$$f(x) = \frac{(x-155)(x-156)(x^2+255)(x^2+258)}{x^2+2018}.$$

If there is, find an interval that contains the number c.

Answers

1) a) The function $f(x)$ is continuous and the interval is closed and bounded. Therefore absolute maximum and minimum values exist.

$$f(x) = \begin{cases} -x+2 & \text{if} \quad x < 4 \\ x-6 & \text{if} \quad x \geqslant 4 \end{cases}$$

$f'(x)$ does not exist at $x = 4$. That's the only critical point.

We have to check critical points and endpoints.

x	$f(x)$
0	2
4	-2
10	4

$\Rightarrow \quad f(4) = -2$ is absolute minimum and $f(10) = 4$ is absolute maximum.

b) This limit is in the form $\dfrac{0}{0}$, so using L'Hôpital's rule we obtain:

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}} \frac{\ln\big(\csc x\big)}{\big(x - \frac{\pi}{2}\big)^2} &= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\csc x}\big(-\cot x \csc x\big)}{2\big(x - \frac{\pi}{2}\big)} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{-\cot x}{2x - \pi} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{\csc^2 x}{2} \\
&= \frac{1}{2}
\end{aligned}$$

2) a) Implicit differentiation with respect to x gives:

$$\frac{1}{\sqrt{1-(xy)^2}}\big(y+xy'\big)+y'\tan\big(\ln x\big)+y\sec^2\big(\ln x\big)\frac{1}{x}=0$$

Insert $x=1,\ y=\dfrac{1}{2}$ to obtain:

$$\frac{\frac{1}{2}+y'}{\sqrt{\frac{3}{4}}}+0+\frac{1}{2}=0$$

$$\Rightarrow \quad y'=-\frac{1}{2}-\frac{\sqrt{3}}{4}$$

b) Using the substitution $\quad u=\sec x \quad \Rightarrow \quad du=\sec x\,\tan x\,dx,$

we obtain $\tan^2 x=u^2-1.$

$$\begin{aligned}\int \tan^5 x\,\sec^6 x\,dx &= \int \tan^4 x\,\sec^5 x\,\sec x\,\tan x\,dx\\ &= \int \big(u^2-1\big)^2 u^5\,du\\ &= \int \big(u^9-2u^7+u^5\big)\,du\\ &= \frac{u^{10}}{10}-\frac{2u^8}{8}+\frac{u^6}{6}+c\\ &= \frac{\sec^{10}x}{10}-\frac{\sec^8 x}{4}+\frac{\sec^6 x}{6}+c\end{aligned}$$

3) a) Using the substitution $u = e^x \quad \Rightarrow \quad du = e^x\,dx,$ we obtain:

$$\int \frac{e^x\,dx}{9+e^{2x}} = \int \frac{du}{9+u^2}$$

$$= \frac{1}{9}\int \frac{du}{1+\left(\frac{u}{3}\right)^2}$$

Now we need a second substitution: $v = \dfrac{u}{3} \quad \Rightarrow \quad dv = \dfrac{1}{3}\,du.$

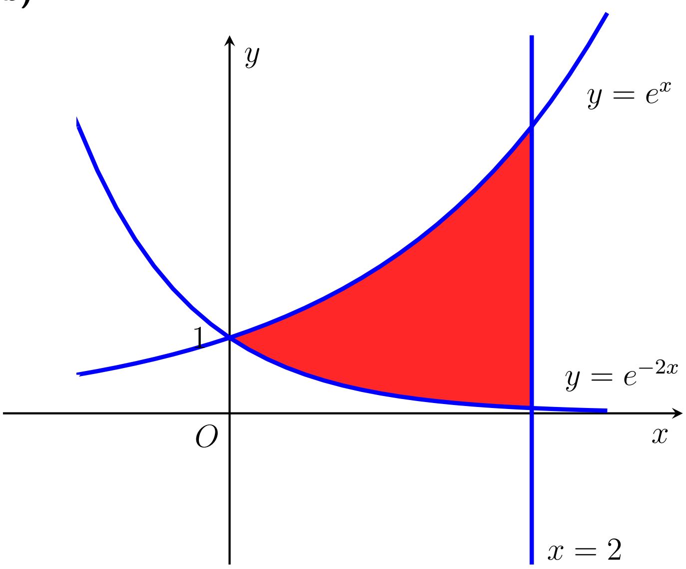
$$= \frac{1}{9}\int \frac{3\,dv}{1+v^2}$$

$$= \frac{1}{3}\arctan v + c$$

$$= \frac{1}{3}\arctan\left(\frac{u}{3}\right) + c$$

$$= \frac{1}{3}\arctan\left(\frac{e^x}{3}\right) + c$$

b)



$$A = \int_0^2 \left(e^x - e^{-2x}\right)dx$$

$$= e^x + \frac{e^{-2x}}{2}\Bigg|_0^2$$

$$= \left(e^2 + \frac{e^{-4}}{2}\right) - \left(1 + \frac{1}{2}\right)$$

$$= e^2 + \frac{e^{-4}}{2} - \frac{3}{2}$$

4) a) Use integration by parts:

$$\begin{aligned} u &= \arcsin(5x) \\ dv &= dx \end{aligned} \quad \Rightarrow \quad \begin{aligned} du &= \frac{5\,dx}{\sqrt{1-25x^2}} \\ v &= x \end{aligned}$$

$$\begin{aligned} \int \arcsin(5x)dx &= x\arcsin(5x) - \int \frac{5x\,dx}{\sqrt{1-25x^2}} \qquad \left(\text{Use } y = 1-25x^2,\ dy = -50x dx\right) \\ &= x\arcsin(5x) + \frac{1}{5}\sqrt{1-25x^2} + c \end{aligned}$$

b)

$$\begin{aligned} I &= \frac{1}{2}\int_0^{1/3} \left[\sin(4\pi x) - \sin(2\pi x)\right] dx \\ &= \frac{1}{2}\left[\frac{-\cos(4\pi x)}{4\pi} + \frac{\cos(2\pi x)}{2\pi}\right]\Bigg|_0^{1/3} \\ &= \frac{1}{4\pi}\left[-\frac{1}{2} + \frac{1}{4} + \frac{1}{2} - 1\right] \\ &= -\frac{3}{16\pi} \end{aligned}$$

c) Let's use the substitution $u = \ln x, \quad du = \frac{dx}{x}.$

$x = 2 \quad \Rightarrow \quad u = \ln 2,$

$x \to \infty \quad \Rightarrow \quad u \to \infty$

$$\begin{aligned} I &= \int_{\ln 2}^{\infty} \frac{du}{u} \\ &= \lim_{R\to\infty} \ln u \Big|_{\ln 2}^{R} \\ &= \lim_{R\to\infty} \ln R - \ln(\ln 2) \\ &= \infty \end{aligned}$$

The integral is divergent.

5) $x^2 - 2x = (x-1)^2 - 1$

$u = x - 1, \quad du = dx$

$$\begin{aligned} I &= \int \frac{(u+1)\,du}{\sqrt{u^2-1}} \\ &= \int \frac{u\,du}{\sqrt{u^2-1}} + \int \frac{du}{\sqrt{u^2-1}} \end{aligned}$$

For the first integral, use the substitution $\quad y = u^2 - 1, \quad dy = 2u\,du.$

For the second integral, use the substitution $\quad u = \sec z, \quad du = \sec z \tan z\,dz.$

$$\begin{aligned} I &= \int \frac{\frac{1}{2}\,dy}{\sqrt{y}} + \int \frac{\sec z \tan z\,dz}{\tan z} \\ &= \frac{1}{2}\,\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + \ln\left|\sec z + \tan z\right| + c \\ &= \sqrt{u^2-1} + \ln\left|u + \sqrt{u^2-1}\,\right| + c \\ &= \sqrt{x^2-2x} + \ln\left|x - 1 + \sqrt{x^2-2x}\,\right| + c \end{aligned}$$

Bonus) $x^2 + 2018 \neq 0 \quad \Rightarrow \quad$ Domain of $f(x)$ is the real line $\mathbb{R}$.

$f(x)$ is continuous and differentiable on its domain.

$\Rightarrow \quad f(x)$ is continuous on $[155,\,156]$ and differentiable on $(155,\,156)$.

Clearly, $f(155) = 0, \quad f(156) = 0.$

Therefore, by Rolle's theorem, there exists a number $c \in (155,\,156)$ such that $f'(c) = 0$.