Çankaya University
Department of Mathematics
2016 - 2017 Fall Semester

MATH 155 - Calculus for Engineering I

First Midterm Examination

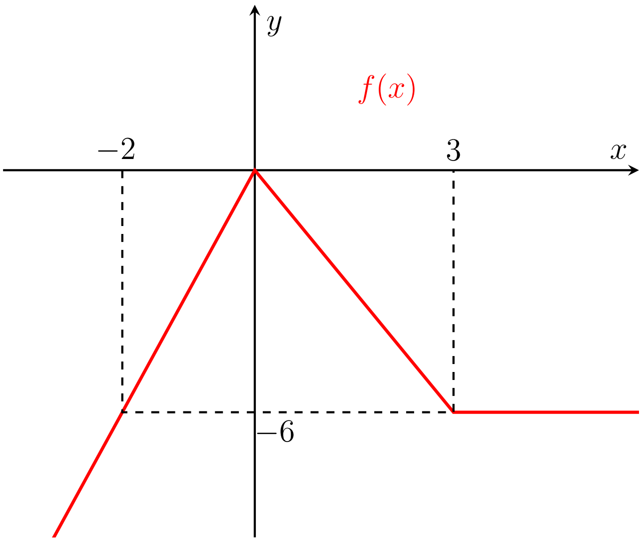
1) Find the domain and range of the following functions. Explain your solution.

a)(5 pts) $f(x) = 3(x-2)^2 + 7$

b)(5 pts) $f(x) = \dfrac{12}{x^4 + 4}$

c)(5 pts) $f(x) = \ln|8x - 7|$

2) (15 pts) Find a formula for the piecewise-defined function in the figure. Explain your solution.



3) Evaluate the following limits if they exist. Explain your solution.
(Do NOT use L'Hôpital's Rule)

a)(5 pts) $\displaystyle\lim_{x\to 0} \frac{x^3 - 27}{x - 3}$

b)(5 pts) $\displaystyle\lim_{x\to 0} \frac{\tan(2x^3)\sin(3x^2)}{x^5 \cos^2 x}$

c)(5 pts) $\displaystyle\lim_{x\to 5} \frac{3x - 15}{|x - 5|}$

4) a)(10 pts) By using Intermediate Value Theorem (IVT), show that the equation

$$x^4 - 9x^2 + 14 = 0$$

has four different roots in the interval $[-3, 3]$. Explain your solution.

b)(10 pts) Find all discontinuities of the following function and classify them as removable or non-removable. Explain your solution.

$$f(x) = \begin{cases} 3x+4 & \text{if} \quad x < 4 \\ 26 & \text{if} \quad x = 4 \\ x^2 & \text{if} \quad 4 < x < 5 \\ 26 & \text{if} \quad x = 5 \\ 52 - x^2 & \text{if} \quad 5 < x < 6 \\ x^2 - 20 & \text{if} \quad x \geq 6 \end{cases}$$

5) Evaluate the following limits if they exist. Explain your solution.
(Do NOT use L'Hôpital's Rule)

a)(5 pts)

$$\lim_{x\to\infty} \sqrt{5x^2 + 3x} - \sqrt{5x^2 + 7x}$$

b)(5 pts)

$$\lim_{x\to\infty} \frac{4 - 2x + x\sqrt{x} - x^{-1} + x^{5/2}}{(13x^2 + 6x + x\sqrt{x})\sqrt{x}}$$

c)(5 pts)

$$\lim_{x\to 2^-} \frac{1}{\ln \frac{x}{2}}$$

6) If exists, find all horizontal, vertical and oblique asymptotes for the following functions. Explain your solution.

a)(10 pts)

$$f(x) = \frac{155x + 2016}{x + 2017}$$

b)(10 pts)

$$f(x) = \frac{3x^3 + x^2 - 20x + 13}{x^2 + x - 6}$$

Answers

1) a) $f(x)$ is a polynomial function so domain$(f) = \mathbb{R}$.
As

$$(x-2)^2 \geqslant 0$$

we have

$$3(x-2)^2 + 7 \geqslant 7$$

so range of f is:

$$[7, \infty)$$

b) As $x^4 + 4 \neq 0$ for any real number, $f(x)$ is defined for all real numbers. So domain$(f) = \mathbb{R}$.
Clearly, $f(x) > 0$ for all $x \in \mathbb{R}$.
Minimum of $x^4 + 4$ is 4 so maximum of f is $\dfrac{12}{4} = 3$.
Therefore range of f is:

$$(0, 3]$$

c) $f(x)$ is defined whenever $|8x - 7| > 0$ so domain$(f) = \mathbb{R} \setminus \{7/8\}$.
Range of $f(x)$ is the same as range of $\ln x$, therefore the range is $\mathbb{R}$.

2) ℓ_1 is a line passing through $(-2, -6)$ and $(0, 0)$. Its slope is

$$m_1 = \frac{0 - (-6)}{0 - (-2)} = 3$$

Its equation is:

$$y - 0 = 3(x - 0) \quad \Rightarrow \quad y = 3x$$

Similarly, ℓ_2 is a line passing through $(3, -6)$ and $(0, 0)$. Its slope is $m_2 = -2$ and its equation is:

$$y = -2x$$

ℓ_3 is a horizontal line with slope zero and equation $y = -6$. Putting all these together, we obtain:

$$f(x) = \begin{cases} 3x & \text{if} \quad x < 0 \\ -2x & \text{if} \quad 0 \leqslant x \leqslant 3 \\ -6 & \text{if} \quad x > 3 \end{cases}$$

3) a) $\lim_{x\to 0} \frac{x^3-27}{x-3} = \lim_{x\to 0} \frac{(x-3)(x^2+3x+9)}{x-3}$

Assuming $x \neq 3$

$= \lim_{x\to 0} (x^2+3x+9)$

$= 9$

b) $\lim_{x\to 0} \frac{\tan(2x^3)\sin(3x^2)}{x^5\cos^2 x} = \lim_{x\to 0} \frac{\tan(2x^3)}{2x^3} \cdot \frac{\sin(3x^2)}{3x^2} \cdot \frac{6}{\cos^2 x}$

$= \lim_{x\to 0} \frac{\sin(2x^3)}{2x^3} \cdot \frac{1}{\cos(2x^3)} \cdot \frac{\sin(3x^2)}{3x^2} \cdot \frac{6}{\cos^2 x}$

$= 1 \cdot 1 \cdot 1 \cdot 6$

$= 6$

c) $\lim_{x\to 5^+} \frac{3x-15}{|x-5|} = \lim_{x\to 5^+} \frac{3x-15}{x-5}$

$= 3$

$\lim_{x\to 5^-} \frac{3x-15}{|x-5|} = \lim_{x\to 5^-} \frac{3x-15}{-x+5}$

$= -3$

Right and left limits are not equal $\Rightarrow$ Limit does NOT exist.

4) a) Let's define the function $f(x) = x^4 - 9x^2 + 14$. Obviously, this is continuous everywhere, because it is a polynomial. Let's check the sign of the function values on certain points:

x	$f(x)$	sign
-3	14	$+$
-2	-6	$-$
-1	6	$+$
0	14	$+$
1	6	$+$
2	-6	$-$
3	14	$+$

Using IVT, we can see that there must be a root to the equation $f(x) = 0$ on $(-3, -2)$, a second root on $(-2, -1)$, a third on $(1, 2)$ and a fourth on $(2, 3)$.

b)We have to check f around 3 points:

$\mathbf{x = 4}$:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 3x + 4 = 16$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x^2 = 16$$

$$\Rightarrow \lim_{x\to 4} f(x) = 16$$

$$f(4) = 26 \neq \lim_{x\to 4} f(x)$$

At $x = 4$, function is discontinuous, because the function value is not equal to the limit. Discontinuity is removable because limit exists.

$\mathbf{x = 5}$:

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} x^2 = 25$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 52 - x^2 = 27$$

$$\Rightarrow \lim_{x\to 5} f(x) \text{ does NOT exist}$$

At $x = 5$, function is discontinuous, because limit does not exist. Discontinuity is non-removable.

$\mathbf{x = 6}$:

$$\lim_{x\to 6^-} f(x) = \lim_{x\to 6^-} 52 - x^2 = 16$$

$$\lim_{x\to 6^+} f(x) = \lim_{x\to 6^+} x^2 - 20 = 16$$

$$\Rightarrow \lim_{x\to 6} f(x) = 16$$

$$f(6) = 16 = \lim_{x\to 6} f(x)$$

At $x = 6$, function is continuous.

5) a) $$\lim_{x\to\infty} \sqrt{5x^2+3x}-\sqrt{5x^2+7x} = \lim_{x\to\infty} \left(\sqrt{5x^2+3x}-\sqrt{5x^2+7x}\right)\cdot\frac{\sqrt{5x^2+3x}+\sqrt{5x^2+7x}}{\sqrt{5x^2+3x}+\sqrt{5x^2+7x}}$$

$$= \lim_{x\to\infty} \frac{5x^2+3x-(5x^2+7x)}{\sqrt{5x^2+3x}+\sqrt{5x^2+7x}}$$

$$= \lim_{x\to\infty} \frac{-4x}{|x|\left(\sqrt{5+\frac{3}{x}}+\sqrt{5+\frac{7}{x}}\right)}$$

$$= \lim_{x\to\infty} \frac{-4}{\sqrt{5+\frac{3}{x}}+\sqrt{5+\frac{7}{x}}}$$

$$= \frac{-2}{\sqrt{5}}$$

b) $$\lim_{x\to\infty} \frac{4-2x+x\sqrt{x}-x^{-1}+x^{5/2}}{(13x^2+6x+x\sqrt{x})\sqrt{x}} = \lim_{x\to\infty} \frac{x^{5/2}\left(1+\frac{1}{x}-\frac{2}{x^{3/2}}+\frac{4}{x^{5/2}}-\frac{1}{x^{7/2}}\right)}{x^{5/2}\left(13+\frac{6}{x}+\frac{1}{x^{1/2}}\right)}$$

$$= \frac{1}{13}$$

c) $x \to 2^-$

$$\Rightarrow \quad \frac{x}{2} \to 1^-$$

$$\Rightarrow \quad \ln\frac{x}{2} \to 0^-$$

$$\Rightarrow \quad \lim_{x\to 2^-} \frac{1}{\ln\frac{x}{2}} = -\infty$$

Limit does NOT exist.

6) a) $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x\left(155+\dfrac{2016}{x}\right)}{x\left(1+\dfrac{2017}{x}\right)} = 155$$

$\Rightarrow \quad y = 155$ is a horizontal asymptote.

$$\lim_{x\to -2017^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to -2017^-} f(x) = \infty$$

$\Rightarrow \quad x = -2017$ is a vertical asymptote.

b)Degree of numerator is 3, degree of denominator is 2, so there is an oblique asymptote. Polynomial division gives:

$$f(x) = 3x - 2 + \frac{1}{x^2 + x - 6} = 3x - 2 + \frac{1}{(x+3)(x-2)}$$

$\Rightarrow \quad y = 3x - 2$ is an oblique asymptote.

$$\lim_{x\to -3^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to -3^-} f(x) = \infty$$

$$\lim_{x\to 2^+} f(x) = \infty \quad \text{and} \quad \lim_{x\to 2^-} f(x) = -\infty$$

$\Rightarrow \quad x = -3$ and $x = 2$ are vertical asymptotes.