Çankaya University
Department of Mathematics
2017 - 2018 Fall Semester

MATH 155 - Calculus for Engineering I

First Midterm Examination

1) a) Find the domain of the function $f(x) = \dfrac{1}{\ln\left(1-2x^2\right)} + \sqrt{x}$. Explain your solution.

b) Sketch the graph of the function $f(x) = \left|x^2 - 6x + 5\right|$.

2) Evaluate the following limits: (Do NOT use L'Hôpital's Rule)

a) $\lim\limits_{x\to\infty} \Big(\ln(x+7) - \ln(x^2+4) \Big)$

b) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{25+3x} - \sqrt{25-7x}}$

c) $\lim\limits_{x\to 0} x^{2018} \sin\left(\dfrac{155}{x}\right)$

3) Evaluate the following limits: (Do NOT use L'Hôpital's Rule)

a) $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{x^2-4}$

b) $\lim\limits_{x\to -\infty} \sqrt{x^2+4x} - \sqrt{x^2+x+1}$

4) Find all discontinuities of $f(x)$. Are they removable or non-removable? Explain.

$$f(x) = \begin{cases} \ln(2-e^x) & \text{if} \quad x < 0 \\ \sin(\pi + x) & \text{if} \quad 0 < x < 1 \\ 1 + e^x & \text{if} \quad x > 1 \end{cases}$$

5) Find all horizontal, vertical and oblique asymptotes of the following functions (if there is any). Explain your solution.

a) $f(x) = (x-3)^2$

b) $f(x) = \dfrac{x^4 - 2x^3 + x^2 + 7x - 9}{x^3 + 8}$

Answers

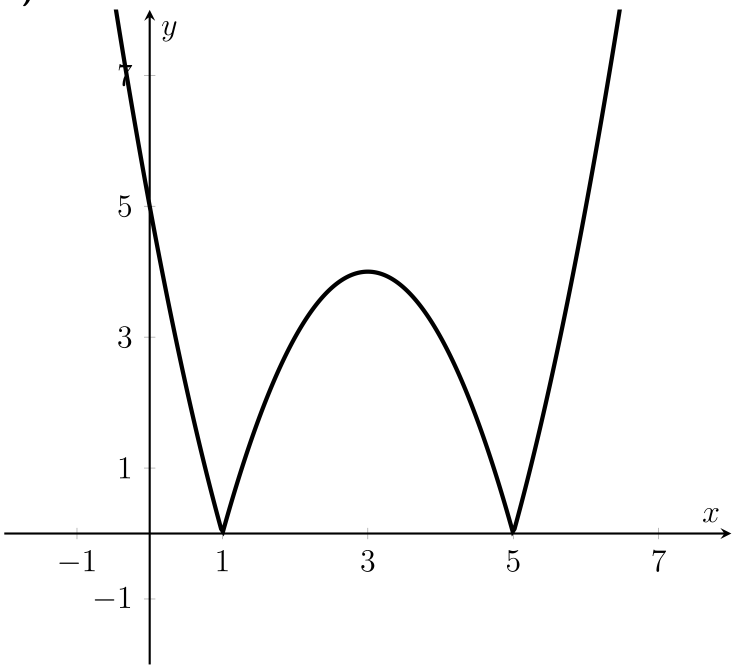
1) a) Domain$\big(\sqrt{x}\big) = [0, \infty)$

$$1 - 2x^2 > 0 \quad \Rightarrow \quad x^2 \leqslant \frac{1}{2}$$

$$\Rightarrow \quad \text{Domain}\left(\frac{1}{\ln\left(1-2x^2\right)}\right) = \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$$

Therefore: Domain$\big(f(x)\big) = \left(0, \dfrac{1}{\sqrt{2}}\right)$

b)



2) a) $\lim\limits_{x\to\infty} \Big(\ln(x+7) - \ln(x^2+4)\Big) = \lim\limits_{x\to\infty} \ln\dfrac{x+7}{x^2+4}$

$$\lim_{x\to\infty} \frac{x+7}{x^2+4} = \lim_{x\to\infty} \frac{1+\frac{7}{x}}{x+\frac{4}{x}} = 0$$

As $x \to \infty$, the ratio $\dfrac{1+\frac{7}{x}}{x+\frac{4}{x}} \to 0^+$ therefore $\lim\limits_{x\to\infty} \ln\dfrac{x+7}{x^2+4} = -\infty$.

b)

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{\sqrt{25+3x}-\sqrt{25-7x}} &= \lim_{x\to 0} \frac{x}{\sqrt{25+3x}-\sqrt{25-7x}} \cdot \frac{\sqrt{25+3x}+\sqrt{25-7x}}{\sqrt{25+3x}+\sqrt{25-7x}} \\
&= \lim_{x\to 0} \frac{x\big(\sqrt{25+3x}+\sqrt{25-7x}\big)}{\big(25+3x\big)-\big(25-7x\big)} \\
&= \lim_{x\to 0} \frac{\sqrt{25+3x}+\sqrt{25-7x}}{10} \\
&= 1
\end{aligned}$$

c) $-1 \leqslant \sin\left(\frac{155}{x}\right) \leqslant 1$

$-x^{2018} \leqslant x^{2018} \sin\left(\frac{155}{x}\right) \leqslant x^{2018}$

$\lim_{x \to 0} -x^{2018} = \lim_{x \to 0} x^{2018} = 0$, therefore

$\lim_{x \to 0} x^{2018} \sin\left(\frac{155}{x}\right) = 0$ by Sandwich Theorem.

3) a) There is an indeterminate form $\left(\frac{0}{0}\right)$.

$$\begin{aligned}
\lim_{x \to 2} \frac{\sin(x-2)}{(x-2)(x+2)} &= \lim_{x \to 2} \frac{\sin(x-2)}{x-2} \cdot \frac{1}{x+2} \\
&= \lim_{x \to 2} \frac{\sin(x-2)}{x-2} \cdot \lim_{x \to 2} \frac{1}{x+2} \\
&= 1 \cdot \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

b) There is an indeterminate form $(\infty - \infty)$.

$$\lim_{x \to -\infty} \left(\sqrt{x^2+4x} - \sqrt{x^2+x+1}\right) \cdot \frac{\sqrt{x^2+4x} + \sqrt{x^2+x+1}}{\sqrt{x^2+4x} + \sqrt{x^2+x+1}}$$

$$\begin{aligned}
&= \lim_{x \to -\infty} \frac{x^2+4x-x^2-x-1}{\sqrt{x^2+4x} + \sqrt{x^2+x+1}} \\
&= \lim_{x \to -\infty} \frac{3x-1}{|x|\left(\sqrt{1+\frac{4}{x}} + \sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)} \\
&= \lim_{x \to -\infty} \frac{3-\frac{1}{x}}{-\left(\sqrt{1+\frac{4}{x}} + \sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)} \\
&= -\frac{3}{2}
\end{aligned}$$

4) $f(x)$ is discontinuous at $x = 0$ and $x = 1$, because it is undefined at these points.

$\lim_{x \to 0^-} \ln(2 - e^x) = 0$ and $\lim_{x \to 0^+} \sin(\pi + x) = 0$ therefore:

$\lim_{x \to 0} f(x) = 0$. So, $x = 0$ is a removable discontinuity.

$\lim_{x \to 1^-} \sin(\pi + x) = \sin(\pi + 1)$ and $\lim_{x \to 1^+} 1 + e^x) = 1 + e$. Obviously, $1 + e > 3$ and left and right limits are different, therefore:

$\lim_{x \to 1} f(x)$ does not exist. So, $x = 1$ is a non-removable discontinuity.

5) a) This is a polynomial, so there's no vertical or oblique asymptotes.

$\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = \infty$ therefore there is no horizontal asymptote.

b) $x = -2$ is a vertical asymptote. There is no horizontal asymptote. Polynomial division gives the oblique asymptote $y = x - 2$.