Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 155 - Calculus for Engineering I

First Midterm Examination

1) a) Find the domain of the function $f(x) = \dfrac{\ln\left(x^3-8\right)}{\sqrt{75-3x^2}} + \dfrac{1}{\left|x-3\right|}$. Explain.

b) Find the range of the function $f(x) = e^{x^2+2x+2}$. Explain.

c) Sketch the graph of the function $f(x) = \begin{cases} -x & \text{if } x < 2 \\ x^2-6 & \text{if } x \geqslant 2 \end{cases}$

2) Evaluate the following limits: (Do NOT use L'Hôpital's Rule)

a) $\displaystyle\lim_{x\to\pi} \frac{\sin(x)-\pi}{x-\pi}$

b) $\displaystyle\lim_{x\to 5} \frac{\frac{6}{x+1}-1}{x-5}$

c) $\displaystyle\lim_{x\to 3} \frac{\left|4x-12\right|}{x^2-9}$

d) $\displaystyle\lim_{x\to 0} \frac{4x^2}{\sqrt{3+x^2}-\sqrt{3-x^2}}$

3) Let the function $f(x)$ be given by: $f(x) = \begin{cases} \dfrac{\sin\left(7x^2\right)}{x\tan\left(3x\right)} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

If it is possible, find the continuous extension of $f(x)$. If it is not possible, explain why not.

4) Using Intermediate Value Theorem, show that the equation $3^x = 5x$ has a root on the interval $\left[0,\, 1\right]$.

5) Find all horizontal, vertical and oblique asymptotes of the following functions (if there is any). Explain your solution.

a) $f(x) = \dfrac{x+1}{x^2-3x-4}$

b) $f(x) = \dfrac{2+3e^x}{e^x-e^{-2x}}$

Answers

1) a) $\ln\left(x^3-8\right)$ is defined when $x^3>8 \quad \Rightarrow \quad$ Domain is: $\left(2,\,\infty\right)$

$\dfrac{1}{\sqrt{75-3x^2}}$ is defined when $75-3x^2>0 \quad \Rightarrow \quad$ Domain is: $\left(-5,\,5\right)$

$\dfrac{1}{\left|x-3\right|}$ is defined when $x\neq 3$.

$\Rightarrow \quad$ Domain of $f(x)$ is intersection of these regions: $\left(2,\,3\right)\cup\left(3,\,5\right)$

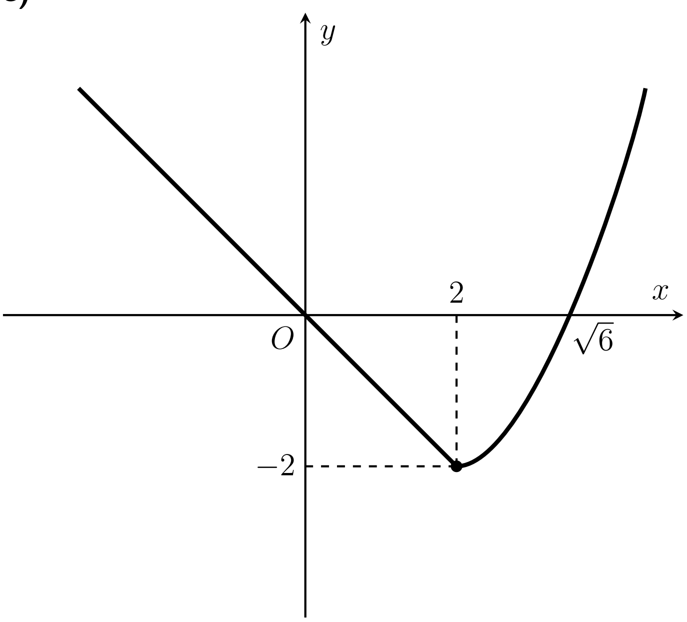
b) $x^2+2x+2=\left(x+1\right)^2+1$

$\left(x+1\right)^2\geqslant 0 \quad \Rightarrow \quad \left(x+1\right)^2+1\geqslant 1$

$\Rightarrow \quad e^{x^2+2x+2}\geqslant e$

Range of $f(x)$ is: $\left[e,\,\infty\right)$

c)



2) a) $\lim_{x\to\pi} x-\pi=0, \quad \lim_{x\to\pi} \sin(x)-\pi=-\pi \quad \Rightarrow \quad \lim_{x\to\pi} \frac{\sin(x)-\pi}{x-\pi}$ does not exist.

b)
$$\lim_{x\to 5} \frac{\frac{6}{x+1}-1}{x-5} = \lim_{x\to 5} \frac{\frac{6}{x+1}-\frac{x+1}{x+1}}{x-5}$$
$$= \lim_{x\to 5} \frac{\frac{5-x}{x+1}}{x-5}$$
$$= \lim_{x\to 5} -\frac{1}{x+1}$$
$$= -\frac{1}{6}$$

c)
$$\lim_{x\to 3^+} \frac{|4x-12|}{x^2-9} = \lim_{x\to 3^+} \frac{4(x-3)}{(x-3)(x+3)}$$
$$= \lim_{x\to 3^+} \frac{4}{x+3}$$
$$= \frac{2}{3}$$

$$\lim_{x\to 3^-} \frac{|4x-12|}{x^2-9} = \lim_{x\to 3^-} \frac{-4(x-3)}{(x-3)(x+3)}$$
$$= \lim_{x\to 3^+} -\frac{4}{x+3}$$
$$= -\frac{2}{3}$$

$$\lim_{x\to 3^+} \frac{|4x-12|}{x^2-9} \neq \lim_{x\to 3^-} \frac{|4x-12|}{x^2-9}$$

Left and right limits are not equal, therefore limit does not exist.

d) $$\lim_{x\to 0} \frac{4x^2}{\sqrt{3+x^2}-\sqrt{3-x^2}} = \lim_{x\to 0} \frac{4x^2}{\sqrt{3+x^2}-\sqrt{3-x^2}} \cdot \frac{\sqrt{3+x^2}+\sqrt{3-x^2}}{\sqrt{3+x^2}+\sqrt{3-x^2}}$$

$$= \lim_{x\to 0} \frac{4x^2\left(\sqrt{3+x^2}+\sqrt{3-x^2}\right)}{\left(3+x^2\right)-\left(3-x^2\right)}$$

$$= \lim_{x\to 0} \frac{4x^2\left(\sqrt{3+x^2}+\sqrt{3-x^2}\right)}{2x^2}$$

$$= \lim_{x\to 0} 2\left(\sqrt{3+x^2}+\sqrt{3-x^2}\right)$$

$$= 2\cdot\left(\sqrt{3}+\sqrt{3}\right)$$

$$= 4\sqrt{3}$$

3) $$\lim_{x\to 0} \frac{\sin\left(7x^2\right)}{x\tan\left(3x\right)} = \lim_{x\to 0} \frac{\sin\left(7x^2\right)}{7x^2}\cdot 7x^2 \cdot \frac{3x}{\tan\left(3x\right)}\cdot\frac{1}{3x}\cdot\frac{1}{x}$$

$$= 1\cdot 1\cdot \lim_{x\to 0}\frac{7x^2}{3x^2}$$

$$= \frac{7}{3}$$

$f(x)$ is discontinuous at $x=0$ because $f(0)=1$ and $\lim_{x\to 0} f(x) = \frac{7}{3}$.

In other words $f(0) \neq \lim_{x\to 0} f(x)$.

Limit exists, therefore the discontinuity is removable. Continuous extension of $f(x)$ is:

$$f_2(x) = \begin{cases} \dfrac{\sin\left(7x^2\right)}{x\tan\left(3x\right)} & \text{if } x\neq 0 \\[2ex] \dfrac{7}{3} & \text{if } x=0 \end{cases}$$

4)

$$f(x) = 3^x - 5x$$

$$f(0) = 3^0 - 0 = 1 \qquad \Big(f(0) > 0\Big)$$

$$f(1) = 3 - 5 = -2 \qquad \Big(f(1) < 0\Big)$$

x	0	1
$f(x)$	$+$	$-$

$f(x)$ is continuous and it changes sign $\Rightarrow$ the equation $f(x) = 0$ has at least one solution in the interval $\big[0,\, 1\big]$ by IVT.

5) a) $\lim\limits_{x\to\infty} f(x) = 0, \quad \lim\limits_{x\to-\infty} f(x) = 0$ therefore the line $y = 0$ $\big(x -$ axis$\big)$ is Horizontal Asymptote.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x+1}{(x+1)(x-4)} = -\frac{1}{5} \text{ (There is no asymptote)}$$

$\lim\limits_{x\to 4^+} f(x) = \infty, \quad \lim\limits_{x\to 4^-} f(x) = -\infty$ therefore the line $x = 4$ is Vertical Asymptote.

There is no Oblique Asymptote because deg$($numerator$)$ $-$ deg$($denominator$) \neq 1$.

b) $\lim\limits_{x\to\infty} f(x) = 3, \quad \lim\limits_{x\to-\infty} f(x) = 0$ therefore the lines $y = 3$ and $y = 0$ are H.A.

$\lim\limits_{x\to 0^+} f(x) = \infty, \quad \lim\limits_{x\to 0^-} f(x) = -\infty$ therefore the line $x = 0$ $\big(y -$ axis$\big)$ is V.A.

There is no Oblique Asymptote because there's no a and b such that $\lim\limits_{x\to\infty} \dfrac{f(x)}{ax+b} = 1$.