Çankaya University
Department of Mathematics
2016 - 2017 Fall Semester

MATH 155 - Calculus for Engineering I

Second Midterm Examination

1) Find the derivatives of the following functions.

a)(5 pts)

$$f(x) = \sin x + \ln x + e^x + 8x^5 + 10\sqrt{x}$$

b)(5 pts)

$$f(x) = \cos^2(4x) + \ln(x^2) + e^{x^3}$$

c)(5 pts)

$$f(x) = x^2 \ln x + e^x \cot x$$

d)(5 pts)

$$f(x) = \frac{xe^x + \ln(3x)}{x^2 + 1}$$

2) a)(10 pts) If y is a function of x, find $\frac{dy}{dx}$:

$$\tan y + x^5 y^2 = 9x^3 + x^{\frac{3}{5}}$$

b)(5 pts) Find $\frac{dy}{dx}$

$$y = \arctan x + \arccos(x^2)$$

3) (20 pts) Find the absolute maximum and absolute minimum of the function

$f(x) = 3^x \sqrt{x^2 - 2}$ on $[-3, -\sqrt{2}]$.

4) (15 pts) Sketch the graph of the function

$$f(x) = \frac{x+3}{2x+4}$$

5) Evaluate the following limits if they exist.

a)(4 pts)

$$\lim_{x \to 0} \frac{1 - \sin(\frac{\pi}{2} + x)}{2x^3 - x}$$

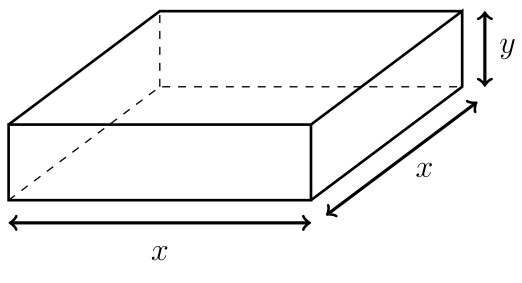
b)(4 pts)

$$\lim_{x \to \infty} \frac{\ln x + x^2}{xe^x}$$

c)(7 pts)

$$\lim_{x \to \infty} \left(1 + \frac{27}{x^2}\right)^x$$

6) (15 pts) An open top box has volume $75\ cm^3$ and is shaped as seen in the figure



Material for base costs $12\ \$/cm^2$ and material for sides costs $10\ \$/cm^2$.
Find the dimensions x and y that give the minimum total cost.

Answers

1) a) $f'(x) = \cos x + \dfrac{1}{x} + e^x + 40x^4 + \dfrac{5}{\sqrt{x}}$

b) $f'(x) = -8\cos(4x)\sin(4x) + \dfrac{2}{x} + 3x^2 e^{x^3}$

c) $f'(x) = 2x\ln x + x + e^x \cot x - \dfrac{e^x}{\sin^2 x}$

d) $f'(x) = \dfrac{\Big(e^x + xe^x + \frac{1}{x}\Big)\cdot(x^2+1) - 2x\cdot\Big(xe^x + \ln(3x)\Big)}{(x^2+1)^2}$

2) a) Using implicit differentiation, we obtain:

$$\sec^2 y\, y' + 5x^4y^2 + 2x^5yy' = 27x^2 + \frac{3}{5}x^{-\frac{2}{5}}$$

$$\Big(\sec^2 y + 2x^5 y\Big)y' = 27x^2 + \frac{3}{5}x^{-\frac{2}{5}} - 5x^4y^2$$

$$y' = \frac{27x^2 + \frac{3}{5}x^{-\frac{2}{5}} - 5x^4y^2}{\sec^2 y + 2x^5 y}$$

b)

$$y' = \frac{1}{1+x^2} - \frac{2x}{\sqrt{1-x^4}}$$

3) $$\begin{aligned} f'(x) &= 3^x \ln 3\sqrt{x^2-2} + 3^x\,\frac{1}{2}(x^2-2)^{-\frac{1}{2}}\,2x \\ &= 3^x\sqrt{x^2-2}\left(\ln 3 + \frac{x}{x^2-2}\right) \end{aligned}$$

Assuming $\ln 3 \approx 1$

$$\begin{aligned} &= 3^x\sqrt{x^2-2}\cdot\frac{x^2+x-2}{x^2-2} \\ &= \frac{3^x\,(x+2)\,(x-1)}{\sqrt{x^2-2}} \end{aligned}$$

Critical points are:

$f' = 0 \quad \Rightarrow \quad x = -2, \quad x = 1$

f' does NOT exist $\quad \Rightarrow \quad x = \pm\sqrt{2}$

We have to check all candidate points, in other words, critical points inside the interval $[-3, -\sqrt{2}]$ and endpoints:

x	$f(x)$
-3	$\frac{\sqrt{7}}{27}$
-2	$\frac{\sqrt{2}}{9}$
$-\sqrt{2}$	0

Using the fact that $3\sqrt{2} > \sqrt{7}$, we obtain:

Absolute maximum is: $f(-2) = \frac{\sqrt{2}}{9}$

Absolute minimum is: $f(-\sqrt{2}) = 0$

4) Domain: $\mathbb{R} \setminus \{-2\}$

$x-$intercept: $y = 0 \quad \Rightarrow \quad x = -3$

$y-$intercept: $x = 0 \quad \Rightarrow \quad y = \frac{3}{4}$

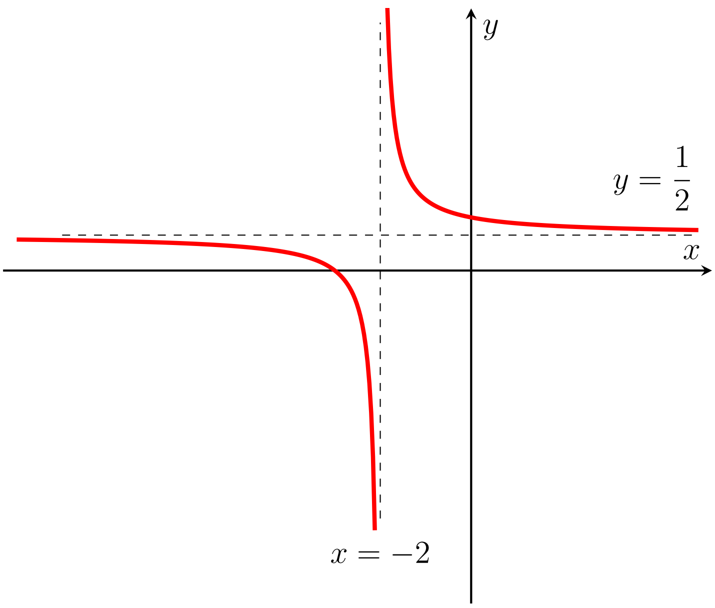
$\lim_{x \to \infty} f(x) = \frac{1}{2}$ and $\lim_{x \to -\infty} f(x) = \frac{1}{2} \quad \Rightarrow \quad y = \frac{1}{2}$ is Horizontal Asymptote.

$\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = \infty \quad \Rightarrow \quad x = -2$ is Vertical Asymptote.

$$f' = \frac{-2}{(2x+4)^2}, \quad f'' = \frac{8}{(2x+4)^3}$$

There's no point where $f' = 0$ or $f'' = 0$. They are undefined at $x = -2$.

x		-2	
f'	$-$	‖	$-$
f''	$-$	‖	$+$
f	$\searrow$	‖	$\searrow$



5) a) This is an indeterminacy of the form $\dfrac{0}{0}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\sin\left(\frac{\pi}{2}+x\right)}{2x^3-x} &= \lim_{x\to 0} \frac{-\cos\left(\frac{\pi}{2}+x\right)}{6x^2-1} \\ &= 0\end{aligned}$$

b) This is an indeterminacy of the form $\dfrac{\infty}{\infty}$ $\Rightarrow$ Use L'Hôpital:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x + x^2}{xe^x} &= \lim_{x\to\infty} \frac{\frac{1}{x}+2x}{e^x+xe^x} \\ &= \lim_{x\to\infty} \frac{-\frac{1}{x^2}+2}{e^x+e^x+xe^x} \\ &= 0\end{aligned}$$

c) This is an indeterminacy of the form 1^∞ $\Rightarrow$ Find the logarithm and use L'Hôpital:

$$L = \lim_{x\to\infty} \left(1+\frac{27}{x^2}\right)^x \quad \Rightarrow \quad \ln L = \lim_{x\to\infty} x \ln\left(1+\frac{27}{x^2}\right)$$

$$\begin{aligned}\lim_{x\to\infty} x\ln\left(1+\frac{27}{x^2}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{27}{x^2}\right)}{\dfrac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{27}{x^2}}\cdot\dfrac{-54}{x^3}}{\dfrac{-1}{x^2}} \\ &= \lim_{x\to\infty} \frac{54x^2}{x^3+27x} \\ &= 0\end{aligned}$$

$$\ln L = 0 \quad \Rightarrow \quad L = 1$$

6) Volume is 75:

$$x^2y = 75 \quad \Rightarrow \quad y = \frac{75}{x^2}$$

Cost function is:

$$\begin{aligned} c(x) &= 12x^2 + 4 \cdot 10xy \\ &= 12x^2 + 40x \cdot \frac{75}{x^2} \\ &= 12x^2 + \frac{3000}{x} \end{aligned}$$

Find the Derivative:

$$\begin{aligned} c'(x) &= 24x - \frac{3000}{x^2} \\ &= \frac{24x^3 - 3000}{x^2} \\ &= \frac{24}{x^2}\left(x^3 - 125\right) \end{aligned}$$

$$c'(x) = 0 \quad \Rightarrow \quad x^3 = 125$$

$$\Rightarrow \quad x = 5 \quad \Rightarrow \quad y = 3$$

Using the fact that $c'(x) < 0$ for $x < 5$ and $c'(x) > 0$ for $x > 5$, we conclude that $x = 5$ gives the local and absolute minimum value.