Çankaya University
Department of Mathematics
2017 - 2018 Fall Semester

MATH 155 - Calculus for Engineering I

Second Midterm Examination

1) Evaluate derivatives of the following functions:

a) $f(x) = \big(155x\big)^{2\ln x}$

b) $f(x) = \tan\Big(\sec^3(\pi x)\Big)$

c) $f(x) = \arctan\big(1 + \sqrt{1+e^x}\big)$

2) Find the equation of the tangent line of $\cos\big(y^2\big) = x^2 - y^2 - 1$ at $\big(\sqrt{\pi}, \sqrt{\pi}\big)$.

3) Evaluate the following limits:

a) $\displaystyle\lim_{x\to\infty}\left(\frac{x+e}{x+\pi^2}\right)^{3x}$

b) $\displaystyle\lim_{x\to\frac{\pi}{4}}\frac{\cos x - \frac{\sqrt{2}}{2}}{4x-\pi}$

c) $\displaystyle\lim_{x\to 0^-}\frac{1}{3^x-1} - \frac{1}{x\ln 3}$

4) Sketch the graph of $f(x) = xe^{1-x^2}$. Show all your work in detail.

5) a) Find the maximum possible value of the product ab where a and b are nonnegative numbers and $3a + 4b = 10$.

b) Find the absolute extrema of the function $f(x) = 2x^3 - 15x^2 + 24x - 12$ on the interval $[2,\ 5]$.

Answers

1) a) $\ln f(x) = \big(2\ln x\big)\ln(155x)$

$$\frac{f'(x)}{f(x)} = \frac{2}{x}\ln(155x) + 2\ln x\left(\frac{155}{155x}\right)$$

$$f'(x) = \big(155x\big)^{2\ln x}\left(\frac{2}{x}\ln(155x) + \frac{2\ln x}{x}\right)$$

b) $f'(x) = \sec^2\big(\sec^3(\pi x)\big)\, 3\sec^2\big(\pi x\big)\,\sec\big(\pi x\big)\,\tan\big(\pi x\big)\pi$

$$= 3\pi\sec^2\big(\sec^3(\pi x)\big)\,\sec^3\big(\pi x\big)\,\tan\big(\pi x\big)$$

c) $f'(x) = \dfrac{1}{1+\big(1+\sqrt{1+e^x}\big)^2}\cdot\dfrac{e^x}{2\sqrt{1+e^x}}$

2) $-\sin(y^2)\cdot 2y\cdot y' = 2x - 2y\cdot y'$

$$y'\Big[2y\big(1-\sin(y^2)\big)\Big] = 2x$$

$$y'\Big|_{\left(\sqrt{\pi},\sqrt{\pi}\right)} = 1$$

Slope of tangent line is 1.

Equation of tangent line is: $y-\sqrt{\pi} = 1\big(x-\sqrt{\pi}\big)$

$\Rightarrow \quad y = x$

3) a)

$$\begin{aligned}\ln L &= \lim_{x\to\infty} 3x\ln\left(\frac{x+e}{x+\pi^2}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(\dfrac{x+e}{x+\pi^2}\right)}{\dfrac{1}{3x}}\end{aligned}$$

Using L'Hôpital's Rule:

$$\begin{aligned}&= \lim_{x\to\infty}\frac{\left(\dfrac{x+\pi^2}{x+e}\right)\left(\dfrac{\pi^2-e}{(x+\pi^2)^2}\right)}{-\dfrac{1}{3x^2}}\\ &= \lim_{x\to\infty}\frac{3x^2\big(e-\pi^2\big)}{(x+e)(x+\pi^2)}\\ &= 3\big(e-\pi^2\big)\end{aligned}$$

$$\Rightarrow \quad L = e^{3(e-\pi^2)}$$

b) This is an indeterminate form of type $\dfrac{0}{0}$ so we will use L'Hôpital's Rule:

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\frac{\sqrt{2}}{2}}{4x-\pi} = \lim_{x\to\frac{\pi}{4}}\frac{-\sin x}{4} = -\frac{\sqrt{2}}{8}$$

c) This is an indeterminate form of type $\infty-\infty$ but we can rewrite it as:

$$\lim_{x\to 0^-}\frac{x\ln 3-3^x+1}{\big(3^x-1\big)\big(x\ln 3\big)}$$

Now it is of type $\dfrac{0}{0}$ so we will use L'Hôpital's Rule:

$$= \lim_{x\to 0^-}\frac{\ln 3-3^x\ln 3}{\big(3^x\ln 3\big)\big(x\ln 3\big)+\big(3^x-1\big)\ln 3}$$

Cancel $\ln 3$, use L'Hôpital's Rule once more:

$$= \lim_{x\to 0^-}\frac{-3^x\ln 3}{x3^x\ln^2 3+3^x\ln 3+3^x\ln 3}$$

$$= -\frac{1}{2}$$

4) Domain of f is $\mathbb{R}$.

$$f'(x) = e^{1-x^2}\left(1 - 2x^2\right)$$

$$f'(x) = 0 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

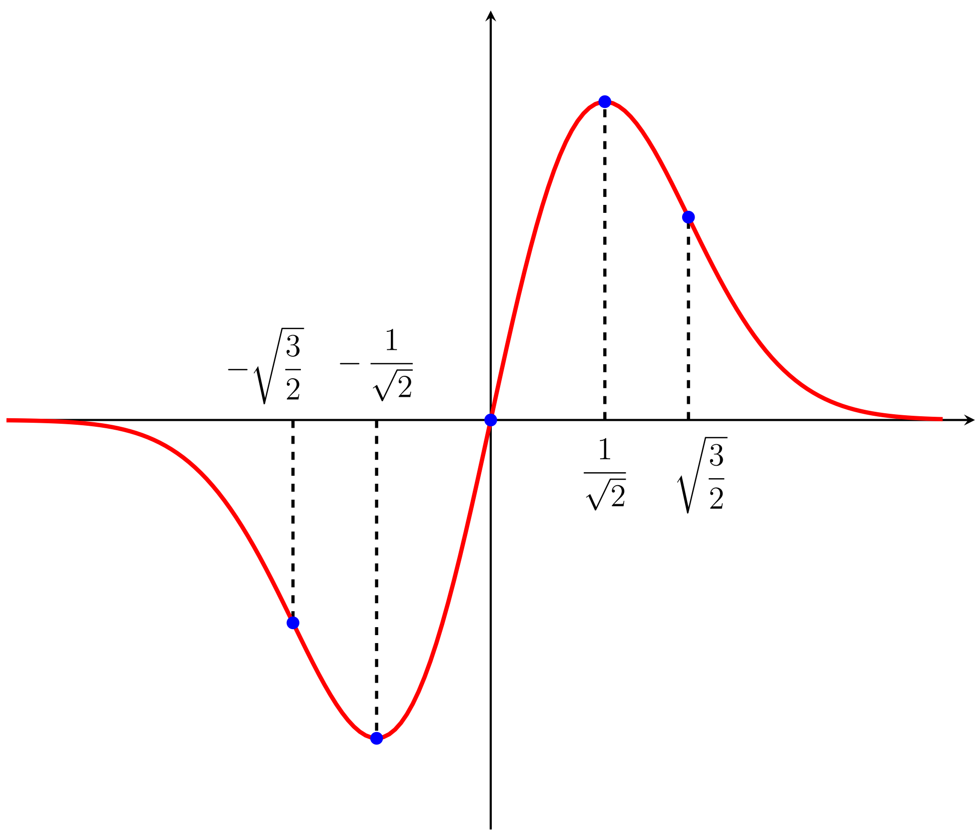
$$f''(x) = -2x\, e^{1-2x^2}\left(3 - 2x^2\right)$$

$$f''(x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = \pm\sqrt{\frac{3}{2}}$$

x		$-\sqrt{\frac{3}{2}}$		$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$		$\sqrt{\frac{3}{2}}$	
f'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
f''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
f	$\searrow$		$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$

Based on the table, we can see that f has a local minimum at $x = -\frac{1}{\sqrt{2}}$, a local maximum at $x = \frac{1}{\sqrt{2}}$ and inflection points on $x = 0, x = \sqrt{\frac{3}{2}}, x = -\sqrt{\frac{3}{2}}$.

There's no vertical or oblique asymptote. $y = 0$ is the horizontal asymptote.



5) a) $a = x$

$$b = \frac{10 - 3x}{4}$$

$$f(x) = x\left(\frac{10 - 3x}{4}\right)$$

$$= \frac{5}{2}x - \frac{3}{4}x^2$$

$$f'(x) = \frac{5}{2} - \frac{3}{2}x = 0$$

$$\Rightarrow \quad x = \frac{5}{3}$$

$$a = \frac{5}{3}, \quad b = \frac{5}{4}$$

$$\Rightarrow \quad ab = \frac{25}{12}$$

b) $f'(x) = 6x^2 - 30x + 24 = 6(x - 1)(x - 4)$

$f'(x) = 0 \quad \Rightarrow \quad x = 1$ or $x = 4.$

$1 \notin [2, 5], \quad 4 \in [2, 5]$

x	f
2	-8
4	-28
5	-17

Absolute maximum: $f(2) = -8,$

Absolute minimum: $f(4) = -28.$