Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 155 - Calculus for Engineering I

Second Midterm Examination

1) a) Find $f'(x)$ where $f(x) = \sin\big(155\arccos(x)\big) + \sqrt[2018]{1+\ln x}$

b) Find $y' = \dfrac{dy}{dx}$ where $y = \left[\dfrac{1+x}{(1+x^3)e^x}\right]^{x^2}$

c) Find all points where the curve $x^2 + xy + y^2 = 6$ has a horizontal tangent and write the equations of these tangent lines.

2) Evaluate the following limits: (if they exist)

a) $\lim\limits_{x\to 0} \dfrac{6e^x - 6 - 6x - 3x^2 - x^3}{x^4}$

b) $\lim\limits_{x\to 0^+} \big(x^2 + x\big)^{\frac{-1}{\ln x}}$

3) Sketch the graph of the function $f(x) = \dfrac{x^2}{x-2}$. Show all your work in detail.

4) a) Mean Value Theorem: Suppose $f(x)$ is a function that is continuous on $\big[a,\, b\big]$ and differentiable on $\big(a,\, b\big)$. Then there is a number $c \in \big(a,\, b\big)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Decide whether the following statement is true or false. Explain your reasons:

$$\big|\sin(7.6) - \sin(7.5)\big| \leqslant 0.1$$

b) Evaluate the limit $\lim\limits_{x\to\infty} \dfrac{e^x + \sin(2x)}{e^{2x} + \ln(3x)}$ if it exists. (This part is unrelated to part **a)**)

5) A helicopter will cover a distance L with constant speed v in time t. The amount of fuel used during flight *per minute* is $a + bv + cv^2$. Given the constants a, b, c, L, find the speed v that minimizes fuel used during *total flight time* t.

Hint: The relationship between speed and time is: $v = \dfrac{L}{t}$.

Answers

1) a) $f'(x) = \cos\big(155 \arccos(x)\big) \cdot 155 \cdot \dfrac{(-1)}{\sqrt{1-x^2}} + \dfrac{1}{2018} \cdot \big(1 + \ln x\big)^{\frac{-2017}{2018}} \cdot \dfrac{1}{x}$

b) $\ln y = x^2 \ln\left[\dfrac{1+x}{(1+x^3)e^x}\right] = x^2\Big(\ln(1+x) - \ln(1+x^3) - x\Big)$

$$\frac{y'}{y} = 2x\Big(\ln(1+x) - \ln(1+x^3) - x\Big) + x^2\left(\frac{1}{1+x} - \frac{3x^2}{1+x^3} - 1\right)$$

$$y' = \left[\frac{1+x}{(1+x^3)e^x}\right]^{x^2}\left[2x\ln(1+x) - 2x\ln(1+x^3) + \frac{x^2}{1+x} - \frac{3x^4}{1+x^3} - 3x^2\right]$$

c) $2x + y + xy' + 2yy' = 0 \quad \Rightarrow \quad y' = \dfrac{-2x-y}{x+2y}$

The slope of a horizontal tangent is zero, therefore $-2x - y = 0 \quad \Rightarrow \quad y = -2x$.

$x^2 - 2x^2 + 4x^2 = 6 \quad \Rightarrow \quad 3x^2 = 6 \quad \Rightarrow \quad x = \pm\sqrt{2}, \quad y = \mp 2\sqrt{2}.$

$y_1 = 2\sqrt{2}, \quad y_2 = -2\sqrt{2}.$

2) a) This is of the form $\dfrac{0}{0} \quad \Rightarrow \quad$ Use L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{6e^x - 6 - 6x - 3x^2 - x^3}{x^4} &= \lim_{x\to 0} \frac{6e^x - 6 - 6x - 3x^2}{4x^3} \\
&= \lim_{x\to 0} \frac{6e^x - 6 - 6x}{12x^2} \\
&= \lim_{x\to 0} \frac{6e^x - 6}{24x} \\
&= \lim_{x\to 0} \frac{6e^x}{24} \\
&= \frac{1}{4}
\end{aligned}$$

b) $L = \lim\limits_{x\to 0^+} \big(x^2 + x\big)^{\frac{-1}{\ln x}} \quad \Rightarrow \quad \ln L = \lim\limits_{x\to 0^+} \dfrac{-1}{\ln x} \ln\big(x^2 + x\big) = \lim\limits_{x\to 0^+} -\dfrac{\ln\big(x^2+x\big)}{\ln x}$

This is of the form $\dfrac{\infty}{\infty} \quad \Rightarrow \quad$ Use L'Hôpital's Rule.

$\ln L = \lim\limits_{x\to 0^+} -\dfrac{2x+1}{x^2+x} \cdot \dfrac{x}{1} = -1$

$\Rightarrow \quad L = e^{-1}$

3) $f(x) = \dfrac{x^2}{x-2} = x + 2 + \dfrac{4}{x-2}$

$y = x + 2$ is Oblique Asymptote.

$\lim\limits_{x \to 2^+} f = \infty$ and $\lim\limits_{x \to 2^-} f = -\infty \quad \Rightarrow \quad x = 2$ is Vertical Asymptote.

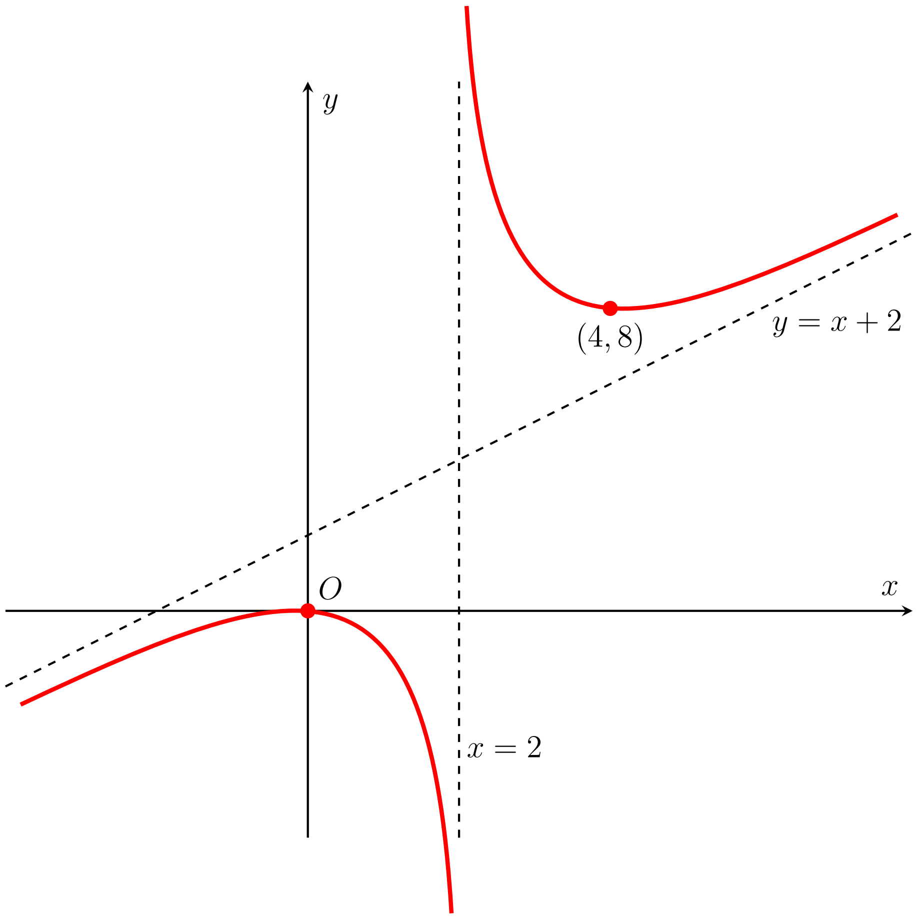
There is no Horizontal Asymptote.

$f' = 1 - \dfrac{4}{(x-2)^2}, \quad f'' = \dfrac{8}{(x-2)^3}$

f'' is never zero. $f' = 0 \quad \Rightarrow \quad x = 4, \quad x = 0$

x		0		2		4	
f'	$+$	0	$-$	‖	$-$	0	$+$
f''	$-$		$-$	‖	$+$		$+$
f	↗		↘	‖	↘		↗

$x = 0$ is a local maximum, $x = 4$ is a local minimum. There is no inflection point.



4) a) Let $f(x) = \sin x$. Then, f satisfies the conditions of MVT. Using $b = 7.6,\ a = 7.5$ we obtain:

$$\frac{\sin(7.6) - \sin(7.5)}{7.6 - 7.5} = \cos(c) \quad \text{where} \quad c \in (7.5,\, 7.6)$$

We know that: $\left|\cos(c)\right| \leqslant 1$

Therefore: $\left|\dfrac{\sin(7.6) - \sin(7.5)}{7.6 - 7.5}\right| \leqslant 1$

$\left|\sin(7.6) - \sin(7.5)\right| \leqslant \left|7.6 - 7.5\right|$

$\left|\sin(7.6) - \sin(7.5)\right| \leqslant 0.1$

So the statement is TRUE.

b) $\displaystyle\lim_{x\to\infty} \frac{e^x + \sin(2x)}{e^{2x} + \ln(3x)} = \lim_{x\to\infty} \frac{1 + \frac{\sin(2x)}{e^x}}{e^x + \frac{\ln(3x)}{e^x}}$

$0 \leqslant \left|\dfrac{\sin(2x)}{e^x}\right| \leqslant \dfrac{1}{e^x} \quad \Rightarrow \quad \displaystyle\lim_{x\to\infty} \frac{\sin(2x)}{e^x} = 0$ by Sandwich theorem.

$\displaystyle\lim_{x\to\infty} \frac{\ln(3x)}{e^x}$ is of the form $\dfrac{\infty}{\infty}$ $\Rightarrow$ Use L'Hôpital's Rule.

$\displaystyle\lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$

Therefore $\displaystyle\lim_{x\to\infty} \frac{1 + \frac{\sin(2x)}{e^x}}{e^x + \frac{\ln(3x)}{e^x}} = 0.$

5) Let $f(v)$ denote the total amount of fuel used during flight.

Clearly, $f(v) =$ (flight time)·(fuel used per unit time), in other words:

$$f(v) = \frac{L}{v}\left(a + bv + cv^2\right) = L\left(\frac{a}{v} + b + cv\right)$$

$$f'(v) = L\left(-\frac{a}{v^2} + c\right), \qquad f'(v) = 0 \quad \Rightarrow \quad v = \sqrt{\frac{a}{c}}$$

$$v \in \big(0,\, \infty\big), \quad \lim_{v\to 0} f(v) = \infty, \quad \lim_{v\to\infty} f(v) = \infty$$

Therefore the minimum of f occurs at the critical point $v = \sqrt{\dfrac{a}{c}}$.

$$f\left(\sqrt{\frac{a}{c}}\right) = L\left(a\sqrt{\frac{c}{a}} + b + c\sqrt{\frac{a}{c}}\right)$$