ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 155 - Calculus for Engineering I

FIRST MIDTERM EXAMINATION

2019-2020 Fall Semester

30.10.2019

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

Question	Grade	Out of
1		15
2		20
3		15
4		20
5		15
6		15
Total		100

IMPORTANT NOTES:

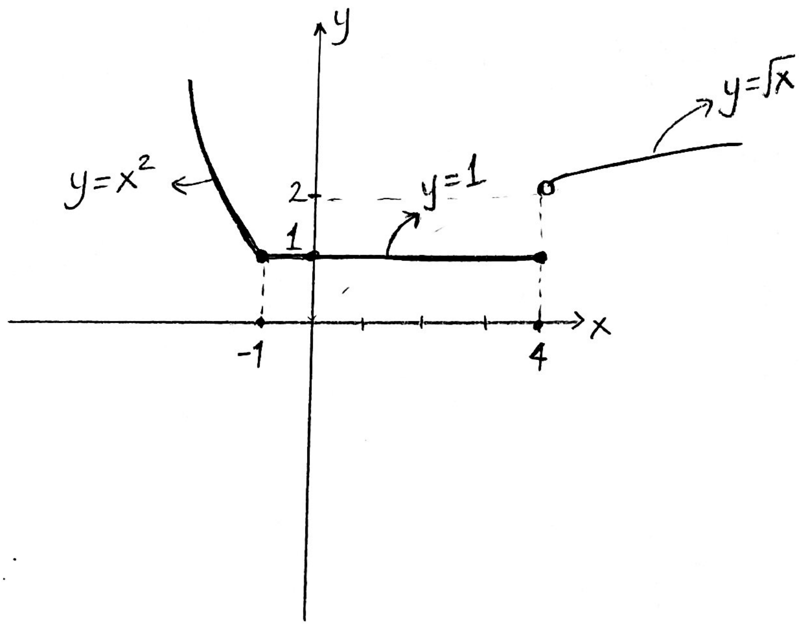
1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Sketch the graph of the following function.

$$f(x) = \begin{cases} x^2 & x < -1, \\ 1 & -1 \le x \le 4, \\ \sqrt{x} & x > 4. \end{cases}$$



2) Evaluate the following limits for the given function, if they exist. Explain your solution. Do not use L'Hôpital's rule.

$$\begin{array}{r|l} x^3+2x^2+4x+3 & x+1 \\ \hline -x^3 \mp x^2 & x^2+x+3 \\ x^2+4x & \\ -x^2 \mp x & \\ 3x+3 & \\ -3x \mp 3 & \\ 0 \quad 0 & \end{array}$$

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^3+2x^2+4x+3} & x < -1, \\[2ex] \dfrac{\sqrt{x+2}-x}{x-2} & -1 < x < 2, \\[2ex] \sqrt{x^2+5x-12}-x & x \geq 2. \end{cases}$$

a) $\lim_{x\to -1} f(x)$.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(\frac{x^2-1}{x^3+2x^2+4x+3}\right) \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to -1}\frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x^2+x+3)} = \frac{-1-1}{1-1+3} = -\frac{2}{3}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\left(\frac{\sqrt{x+2}-x}{x-2}\right) = \frac{\sqrt{-1+2}-(-1)}{-1-2} = \frac{1+1}{-3} = -\frac{2}{3}$$

$$\Rightarrow \boxed{\lim_{x\to -1} f(x) = -\frac{2}{3}}$$

b) $\lim_{x\to 2} f(x)$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{\sqrt{x+2}-x}{x-2}\right) \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 2^-}\frac{(x+2-x^2)}{(x-2)(\sqrt{x+2}+x)} = \lim_{x\to 2^-}\frac{-\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(\sqrt{x+2}+x)} = \frac{-3}{4}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(\sqrt{x^2+5x-12}-x\right) = \sqrt{2}-2$$

$$\Rightarrow \lim_{x\to 2} f(x) = \text{D.N.E.}$$

c) $\lim_{x\to -\infty} f(x)$.

$$= \lim_{x\to -\infty}\left(\frac{x^2-1}{x^3+2x^2+4x+3}\right) = \lim_{x\to -\infty}\frac{\cancel{x^3}\left(\frac{1}{x}-\frac{1}{x^3}\right)}{\cancel{x^3}\left(1+\frac{2}{x}+\frac{4}{x^2}+\frac{3}{x^3}\right)} = \frac{-0+0}{1-0+0-0} = \frac{0}{1} = \boxed{0}$$

d) $\lim_{x\to \infty} f(x)$.

$$= \lim_{x\to\infty}\left(\sqrt{x^2+5x-12}-x\right) = \lim_{x\to\infty}\frac{(\cancel{x^2}+5x-12)-\cancel{x^2}}{(\sqrt{x^2+5x-12}+x)} = \lim_{x\to\infty}\frac{(5x-12)}{(\sqrt{x^2+5x-12}+x)}$$

$$= \lim_{x\to\infty}\left[\frac{\cancel{x}\left(5-\frac{12}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{5}{x}-\frac{12}{x^2}}+1\right)}\right] = \frac{5-0}{\sqrt{1+0-0}+1} = \frac{5}{2}$$

3) Find the domain of the following functions. Explain your solution.

a) $f(x) = \frac{1}{|x^2 - x - 1|}$.

i) $x^2 - x - 1 \neq 0 \Rightarrow \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 1 \neq 0 \Rightarrow \left(x - \frac{1}{2}\right)^2 \neq \frac{5}{4} \Rightarrow x - \frac{1}{2} \neq \mp \frac{\sqrt{5}}{2}$

$\Rightarrow x \neq \mp \frac{\sqrt{5}}{2} + \frac{1}{2}$

$x \neq \frac{\sqrt{5}+1}{2}$ and $x \neq \frac{-\sqrt{5}+1}{2}$

$$\boxed{\text{Dom.: } \left(-\infty, \frac{-\sqrt{5}+1}{2}\right) \cup \left(\frac{-\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}\right) \cup \left(\frac{\sqrt{5}+1}{2}, \infty\right)}$$

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b) $f(x) = \sqrt{\ln(\ln(x))}$.

i) $\ln(\ln(x)) \geq 0 = \ln 1 \Rightarrow \ln x \geq 1 = \ln e \Rightarrow x \geq e$

ii) $\ln(x) > 0 = \ln 1 \Rightarrow x > 1$

iii) $x > 0$

$x \geq e$ satisfies all 3 requirements

$\Rightarrow \boxed{\text{Dom } f: [e, \infty)}$

4) This question has two unrelated parts.
a) Given $f(x) = x^3 + x + 1$, verify by using Intermediate Value Theorem that there is a point $c \in [-1, 0]$ such that $f(c) = 0$.

$f(x) = x^3 + x + 1$

$f(-1) = -1 - 1 + 1 = -1 < 0$

$f(0) = 0 + 0 + 1 = 1 > 0$

Since $f(x) = x^3 + x + 1$ is continuous on $[-1, 0]$ and since $f(-1) < 0$, $f(0) > 0$

$\Rightarrow$ by Int. Val. Thm. $\exists\, c \in (-1, 0)$ s.t. $f(c) = 0$

b) Examine the continuity of $e^{\frac{1}{1-x}}$. Find the point where the function is discontinuous and classify it as removable or non-removable.

$f(x) = e^{\frac{1}{1-x}}$

$\lim_{x \to 1^-} e^{\frac{1}{1-x}} = e^{\frac{1}{0^+}} = e^{\infty} = \boxed{\infty}$

$\begin{pmatrix} x < 1 \\ x - 1 < 0 \\ 1 - x > 0 \end{pmatrix}$

$\lim_{x \to 1^+} e^{\frac{1}{1-x}} = e^{\frac{1}{0^-}} = e^{-\infty} = \frac{1}{e^{\infty}} = \boxed{0}$

$\begin{pmatrix} x - 1 > 0 \\ 1 - x < 0 \end{pmatrix}$

Since $\lim_{x \to 1^-} f(x) = \infty$ (D.N.E.) $\Rightarrow$ discontinuity at $x = 1$ is non-removable.

5) Evaluate the following limits for the given function, if they exist. Explain your solution. Do not use L'Hôpital's rule.

a) $\lim_{x\to 0} x^2 \cos(\frac{1}{\sqrt{x}})$. Since $\frac{1}{\sqrt{x}}$ is defined for $x>0 \Rightarrow$

for $\underline{\lim_{x\to 0^+} x^2 \cdot \cos\left(\frac{1}{\sqrt{x}}\right)}$: $-1 \le \cos\left(\frac{1}{\sqrt{x}}\right) \le 1 \Rightarrow -x^2 \le x^2 \cdot \cos\left(\frac{1}{\sqrt{x}}\right) \le x^2$

$$\Rightarrow \underbrace{\lim_{x\to 0^+}(-x^2)}_{\substack{\parallel \\ 0}} \le \underbrace{\lim_{x\to 0^+}\left(x^2 \cdot \cos\left(\frac{1}{\sqrt{x}}\right)\right)}_{} \le \underbrace{\lim_{x\to 0^+}(x^2)}_{\substack{\parallel \\ 0}}$$

By Squeeze (Sandwich) Law $\Rightarrow \boxed{\lim_{x\to 0^+}\left(x^2 \cdot \cos\left(\frac{1}{\sqrt{x}}\right)\right) = 0}$

Note that; since $\frac{1}{\sqrt{x}}$ is not defined for $x<0 \Rightarrow \boxed{\lim_{x\to 0^-}\left(x^2 \cdot \cos\left(\frac{1}{\sqrt{x}}\right)\right) = \text{d.n.e.}}$

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b) $\lim_{x\to 1} \frac{x^{3/5} - 1}{1 - x^{5/3}}$.

$x^{1/15} = u \Rightarrow x^{3/15} = u^3 \Rightarrow x^{3/5} = (u^3)^3 = u^9$

$x^{3/15} = x^{1/5}$

as $x\to 1$, $u \to 1$

$x^{5/15} = u^5 \Rightarrow x^{5/3} = (u^5)^5 = u^{25}$

$x^{5/15} = x^{1/3}$

$$= \lim_{u\to 1}\left(\frac{u^9 - 1}{1 - u^{25}}\right) = \lim_{u\to 1} \frac{\cancel{(u-1)}\overbrace{(u^8 + u^7 + u^6 + \cdots + 1)}^{9\text{-terms}}}{\underset{(-1)}{\cancel{(1-u)}}\underbrace{(1 + u + u^2 + \cdots + u^{24})}_{25\text{-terms}}} = \frac{9}{-25}$$

6) If exist, find all horizontal, vertical and oblique asymptotes of the following function. Explain your solution.

$$\frac{x^2-4}{x} = x - \frac{4}{x}$$

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^{\frac{1}{x-1}} & x < 0, \\ \frac{x^2-4}{x} & x > 0. \end{cases}$$

$\underline{x>0:}$ $\lim_{x\to\infty} \left(\frac{x^2-4}{x} - x\right) = \lim_{x\to\infty} \left(-\frac{4}{x}\right) = 0 \Rightarrow$ $\boxed{y=x}$ is $\underline{\text{oblique asymptote}}$ as $x\to\infty$

$\underline{x<0:}$ $\lim_{x\to-\infty} \left(\frac{1}{2}\right)^{\frac{1}{x-1}} = \left(\frac{1}{2}\right)^{\lim_{x\to-\infty}\left(\frac{1}{x-1}\right)} = \left(\frac{1}{2}\right)^{\left(\frac{1}{-\infty}\right)} = \left(\frac{1}{2}\right)^{0} = 1$

$\Rightarrow$ $\boxed{y=1}$ is $\underline{\text{horizontal asymptote}}$ as $x\to-\infty$

$\lim_{\substack{x\to0^+ \\ (x>0)}} f(x) = \lim_{x\to0^+} \left(\frac{x^2-4}{x}\right) = \lim_{x\to0^+} \left(x - \frac{4}{x}\right) = 0 - \infty = \boxed{-\infty}$

$\Rightarrow$ there is $\underline{\text{vertical asymptote}}$ at $\underline{x=0}$.