1) Evaluate the derivatives of the following functions.

a) $f(x) = e^x \ln x + x^3$.

$$f'(x) = e^x . \ln x + e^x . \frac{1}{x} + 3x^2$$

b) $f(x) = \sin(\cos(x^2))$.

$$f'(x) = (\cos(\cos(x^2))) . (-\sin(x^2) .)(2x)$$

c) $f(x) = \dfrac{\ln(x^2+1)}{x^3+4x^2}$.

$$f'(x) = \frac{\left(\frac{2x}{x^2+1}\right)(x^3+4x^2) - (3x^2+8x)\ln(x^2+1)}{(x^3+4x^2)^2}$$

$$= \frac{(2x)(x^3+4x^2) - (x^2+1)(3x^2+8x)\ln(x^2+1)}{(x^2+1)(x^3+4x^2)^2}$$

2) Evaluate the following limits (if they exist).

a) $\lim_{x\to 0} \frac{\ln\left(\frac{1+2x}{1-2x}\right)}{x} = \frac{0}{0} \Rightarrow$ use L'Hôpital's rule

Instead, write the limit as: $\lim_{x\to 0} \frac{\ln(1+2x) - \ln(1-2x)}{x} = \frac{0}{0}$

$$\overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{\frac{2}{1+2x} - \frac{-2}{1-2x}}{1} = \frac{2+2}{1} = 4$$

b) $\underbrace{\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x}_{L} = 1^\infty$ (indeterminate form)

$$\ln L = \lim_{x\to\infty} x \cdot \ln\left(1+\frac{3}{x}\right) = \infty \cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \frac{0}{0} \Rightarrow \text{(L'Hôpital's rule)}$$

$$= \lim_{x\to\infty} \frac{\frac{-\frac{3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\left(-\cancel{3/x^2}\right)\cdot\left(-\cancel{x^2}\right)}{1+3/x} = \frac{3}{1+0} = 3$$

$$\Rightarrow \ln L = 3 \Rightarrow L = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = e^3$$

3) Let

$$f(x) = \begin{cases} x^3 - 3x^2 - 9x + 7 & x \le 0, \\ x^2 - 4x + 7 & x > 0. \end{cases}$$

a) Write an equation of the tangent line to the curve $y = f(x)$ at $x = 3$.
b) Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-2, 4]$.

$$f'(x) = \begin{cases} 3x^2 - 6x - 9, & x < 0 \\ 2x - 4, & x > 0 \end{cases}$$

at $x=3 \Rightarrow f'(3) = 2(3) - 4 = 2 \Rightarrow$ slope $= 2$

at $x=3 \Rightarrow f(3) = 3^2 - 4(3) + 7 = 4$

$\Rightarrow$ tg. line at $(3,4)$: $y - 4 = 2(x-3)$

$\Rightarrow y = 2x - 6 + 4 \Rightarrow \boxed{y = 2x - 2}$

b) $f'(x) = \begin{cases} 3x^2 - 6x - 9 = 0 \Rightarrow 3(x-3)(x+1) = 0 \\ \cancel{x=3},\ x = -1 \rightarrow, x \le 0 \\ 2x - 4 = 0 \Rightarrow x = 2 \text{ satisfies } x > 0 \end{cases}$

$f'(0)$ d.i.n.e.

$\boxed{\text{Crit. pts: } x = -1 \ \& \ x = 2 \ \& \ x =}$

x	f(x)
−2	$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 7 = 5$
−1	$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = 12$ → Abs. max.
2	$f(2) = (2)^2 - 4(2) + 7 = 3$ → Abs. min.
4	$f(4) = (4)^2 - 4(4) + 7 = 7$
0	$f(0) = 7$

end pts

4) Evaluate $\frac{dy}{dx}$.

a) $y^x = x^{y^2}$. $\Rightarrow \ln y^x = \ln x^{y^2} \Rightarrow x \ln y = y^2 . \ln x$

$$\frac{d}{dx}: \left(1 . \ln y + x . \frac{y'}{y}\right) = 2y . y' . \ln x + y^2 . \frac{1}{x}$$

$$\Rightarrow y'\left[\frac{x}{y} - 2y . \ln x\right] = \frac{y^2}{x} - \ln y$$

$$y' = \frac{\frac{y^2}{x} - \ln y}{\frac{x}{y} - 2y . \ln x} = \boxed{\frac{y^3 - xy . \ln y}{x^2 - 2xy^2 \ln x}}$$

b) $y = \log_x \sin x$. $= \frac{\ln \sin x}{\ln x}$

$$y' = \frac{\left(\frac{\cos x}{\sin x}\right)(\ln x) - \left(\frac{1}{x}\right)(\ln \sin x)}{(\ln x)^2}$$

$$= \boxed{\frac{(x)(\cos x)(\ln x) - (\sin x)(\ln \sin x)}{(x)(\sin x)(\ln x)^2}}$$

5) Sketch the graph of the curve $f(x) = (x-1)^2e^{2x}$.
(Indicate the domain of the function, $x-$ and $y-$ intercepts and local extrema.)

$f(x) = (x-1)^2 e^{2x} \Rightarrow$ Domain $f: \mathbb{R}$

$f(0) = (0-1)^2 \cdot e^{2(0)} = 1 \Rightarrow \boxed{(0,1) \rightarrow \text{y-intercept}}$

$y = 0 \Rightarrow \underset{e^{2x} \neq 0}{x = 1} \Rightarrow \boxed{(1,0) \rightarrow \text{x-intercept}}$

$f'(x) = 2(x-1)e^{2x} + 2(x-1)^2 \cdot e^{2x} = 2(x-1)e^{2x}\underbrace{[1+(x-1)]}_{x} = 0$

$\Rightarrow \boxed{f'(x) = 2(x-1)e^{2x}(x)}$

$\Rightarrow$ crit. pts : $\boxed{x=1 \text{ and } x=0}$ $\boxed{f(1)=0\ ,\ f(0)=1}$

$f''(x) = 2[1 \cdot e^{2x} \cdot x + (x-1) \cdot 2e^{2x} \cdot x + (x-1)e^{2x} \cdot 1] = \boxed{2e^{2x}(2x^2-1)}$

$f''(x) = 0 \Rightarrow \boxed{x = \mp \frac{1}{\sqrt{2}}} \rightarrow$ inf. pts. $f\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{3-2\sqrt{2}}{2}\right)e^{\sqrt{2}}$

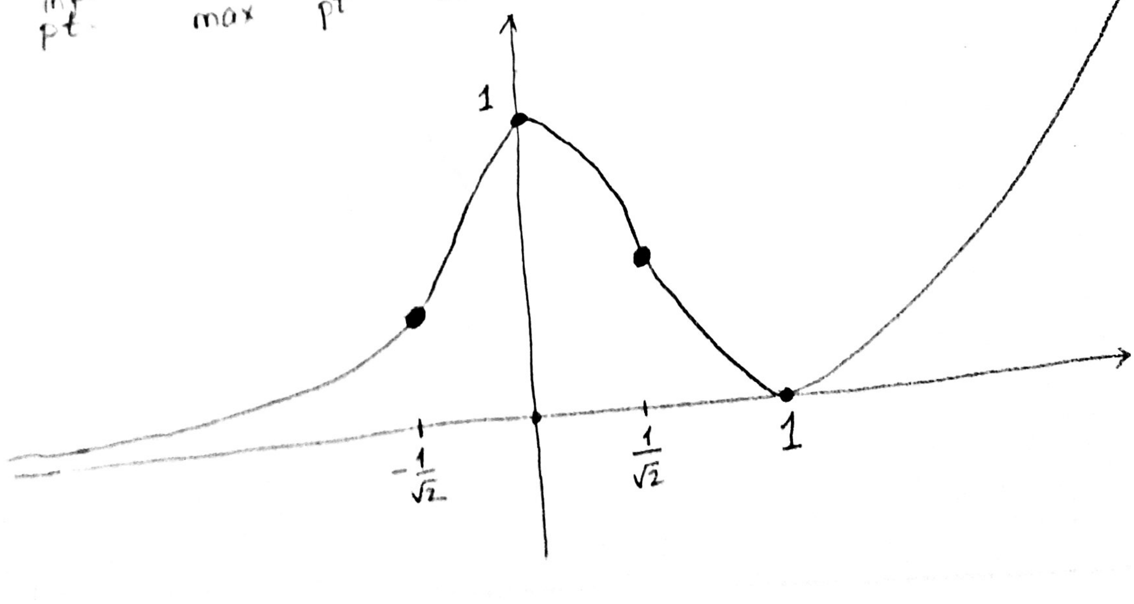
$f\left(-\frac{1}{\sqrt{2}}\right) = \left(\frac{3+2\sqrt{2}}{2}\right)e^{-\sqrt{2}}$

$\lim\limits_{x \to \infty} f(x) = \infty\ ,\ \lim\limits_{x \to -\infty} f(x) = 0$

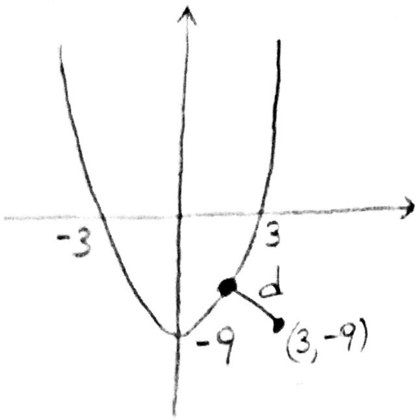
x		$-1/\sqrt{2}$		0		$1/\sqrt{2}$		1	
f'	+		+	0	—		—	0	+
f''	+	0	—		—	0	+		+
f									
		inf pt		loc. max		inf pt		loc. min	

Local extrema: $(0,1) \rightarrow$ loc. max
$(1,0) \rightarrow$ loc. min

Inf. pts : $\left(\frac{1}{\sqrt{2}}, \left(\frac{3-2\sqrt{2}}{2}\right)e^{\sqrt{2}}\right)$
$\left(-\frac{1}{\sqrt{2}}, \left(\frac{3+2\sqrt{2}}{2}\right)e^{-\sqrt{2}}\right)$



6) Find the point on the parabola $y = x^2 - 9$ closest to the point $(3, -9)$.



$$d = \sqrt{(x-3)^2 + ((\underbrace{x^2-9}_{y}) - (-9))^2}$$

$$d^2 = (x-3)^2 + x^4$$

$$\frac{d}{dx}(d^2) = 2(x-3) + 4x^3 = 0$$

$$\Rightarrow 2\left[2x^3 + x - 3\right] = 0$$

$$(x-1)\underbrace{(2x^2+2x+3)}_{\substack{\Delta = (2)^2 - 4(2)(3) < 0 \\ \text{no real roots.}}} = 0$$

$\Rightarrow$ only critical value is $x = \underline{1}$

$x = 1 \Rightarrow y = 1^2 - 9 = -8 \Rightarrow \boxed{(1,-8)} \rightarrow$ closest point to $(3,-9)$ on $y = x^2 - 9$

x		1
$\frac{d}{dx}(d^2)$	$-$	$+$
d^2	$\searrow$	$\nearrow$

$\hookrightarrow$ min. pt.